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HEAVISIDE'S OPERATIONAL CALCULUS

ELECTRICAL ENGINEERING TEXTS

HEAVISIDE'S OPERATIONAL CALCULUS

AS APPLIED TO ENGINEERING AND PHYSICS

BY

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PREFACE

The increasing importance of studying transient phenomena in electrical transmission and machinery is responsible for the present interest in Heaviside's operational mathematics. The author has been demonstrating Heaviside's methods for several years and has invariably found great interest on the part of students. At first, a course was given to Graduate students only, later to Senior electrical engineers, and within the last few years to Juniors. Experience indicates that after Juniors have become familiar with the physical aspects of self-inductance, capacitance, and mutual inductance, they can advantageously be introduced to Heaviside's work and be shown that the very important representation of vectors by complex numbers is only a special case of Heaviside's operational solution.

The greater part of Heaviside's work is to be found in his three volumes on "Electromagnetic Theory" and in two volumes of his *Electrical Papers*.

The first volume of "Electromagnetic Theory" deals largely with vector analysis. This phase of his work is easily understood and will not be discussed here, as a knowledge of vector analysis is not necessary in order to understand his operational mathematics, introduced in the second volume. There, unfortunately, he plunges into problems leading to fractional integration and differentiation, which necessitate the most cautious use of the operational method. Indeed, he begins by practically admitting that certain of his mathematical transformations are not so evident as they seem and that great caution must be exercised. These and similar reservations do not encourage the study of his work. Again, when he introduces the important expansion theorem, he apparently considers it too self-evident to require a proof. It is unfortunate that he did not devote a few pages to discussing his *unit function*, the foundation of his whole structure. It would be highly interesting and instructive to learn just how he arrived at his method. Afterthought is much easier than forethought, and it is now possible to see several ways by which he may have arrived at his result.

Since the present book deals only with a very limited field of Heaviside's work—the application of his operational calculus—and since his pioneer work in other directions is just as important and, in some respects, more so, it has seemed desirable to include in an appendix a most interesting summary of Heaviside's work, as well as something about his personality, published by Mr. B. A. Behrend in the *Electric Journal* for January and February, 1928.

I am greatly indebted to Mr. Behrend for the use of this article, to Mr. S. J. Haefner, who has written Chap. XXVIII, dealing with the solution of transcendental equations frequently occurring in the book, and to Dr. J. J. Smith and Mr. S. J. Haefner for valuable suggestions in the preparation of the book. I am also indebted to Dr. F. W. Grover for assistance in proof reading and to Mr. John R. Carson, and Dr. Vannevar Bush who have kindly permitted me to refer to the bibliographies which they have compiled.[1]

A great part of this present book has appeared in a series of articles printed in the *General Electric Review*, beginning with the number for December, 1927, and ending with that for September, 1928. I am indebted to the editor of the *Review*, Mr. E. C. Sanders, for his personal interest in these articles and for permission to have them reprinted in book form. Last but not least I am indebted to Dr. Harry E. Clifford of Harvard Unversity who has been of great help in editing the book.

E. J. B.

SCHENECTADY, N. Y.
July, 1929.

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[1] CARSON. "Electric Circuit Theory and the Operational Calculus." McGraw-Hill Book Company, Inc.

BUSH. "Operational Circuit Analysis." John Wiley and Son Inc.

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BRIEF PERSONAL SKETCH OF OLIVER HEAVISIDE

Oliver Heaviside was born in London on May 13, 1850, and died at Torquay on Feb. 3, 1925. His death was unexpected and was probably the indirect result of a fall from a ladder in November, 1924. For many years he lived alone, apparently quite isolated from family and friends, his principal contact with the world being through his "Bobby"—a policeman who bought and delivered his supplies.

At the time of his death he was a strikingly handsome old man with snow-white beard and hair, keen, friendly eyes, and delicate hands. He was of medium height with only a slight suggestion of stooping. Although his hearing was defective, he was far from deaf, and it was not difficult to converse with him. His eyesight was remarkable; he read with ease the smallest print. But his general health was not good, and this he attributed largely to suffering from lack of heat and proper food during the World War. Nevertheless, he rarely complained. He was an optimist and counted on living to a very old age.

He was exceedingly kind and generous and exceptionally gracious in his manner. When once privileged to know him—and the author was fortunate enough to spend several days with him—one could not help loving him. His home—on top of a hill—overlooked the bay at Torquay, and it would be difficult to find a more beautiful view in any country. Heaviside admired it and often referred to it in conversation. He had the soul of an artist; and he was indeed an artist, though he never applied himself to painting. His father had had considerable talent, to judge by an oil painting of little Oliver climbing a fence in a pasture.

Living alone, it was natural that Heaviside's home should not be so well kept up as it might have been. Yet considering the circumstances, it was really remarkably clean. The windows were immaculate. I noticed upon my second and subsequent visits that he had put newspapers over the stair carpets. Evidently it was less work to spread the papers than to wipe off the dust that I might have brought in.

It was apparent that he took pride in his personal appearance, for while his suit may not have been of the latest cut, it was always spotless. He might well have served as model of the gentleman scholar of the olden days.

While Heaviside's chief concern was along scientific lines and he has been ranked with Newton, Archimedes, Kelvin, and Faraday, he was a man of many other interests as well. He was very well versed in literature and delighted in quoting Shakespeare and Shaw. The walls of his study were covered with pictures of famous men in all fields, and he took delight in speaking of them.

Little is known of his history, as he was exceedingly reluctant to speak about himself and had evidently requested his brother, who also lived in Torquay and whom he survived by only a few weeks, not to make public any facts about his career. To the author, however, scheduled for another visit in June, 1925, he had promised not only a photograph but also some details of his life, and he would have kept his word; he referred in several letters to the expected visit.

He was a prolific writer of letters as well as of scientific articles. His penmanship was exceptional, and his letters were not only highly interesting but also artistically beautiful. He made his own pen and ink, because "with any commercial pen, the ink will not flow uniformly."

In a tribute to Heaviside in *Annales des postes, télégraphes et Téléphones,* June, 1925, Bethenod gives a very interesting sketch of his contributions to science as well as some facts about his life. From this and other sources[1] it appears that he was connected until 1874 with the Great Northern Telegraph Company at Newcastle upon Tyne. He was a nephew of the famous telegraph engineer Sir Charles Wheatstone and probably was influenced by him in choosing his career.

---

[1] Tributes by Sir Oliver Lodge and B. A. Behrend, *Elec. World,* Feb. 21, 1925. LODGE, SIR OLIVER, *Electrician,* Vol. XCIV, p. 174. RUSSELL, DR. ALEXANDER, *Nature,* Vol. 115, p. 237. GILL, F., *Bell System Tech. Jour.,* July, 1925. BEHREND, B. A., *Elec. Jour.,* January and February, 1928.

He had at least a fair elementary school education. In mathematics he studied algebra and trigonometry in school but apparently nothing beyond. He made some contributions to electrical literature at twenty and began his own serious mathematical preparation in 1873, when he decided to learn to read Maxwell's "Electromagnetic Theory." He practically ceased his scientific work in 1912 at the age of sixty-two.

HEAVISIDE'S OPERATIONAL CALCULUS

AS APPLIED TO ENGINEERING AND PHYSICS

CHAPTER I

HEAVISIDE'S "UNIT FUNCTION" AND THE ALGEBRAIC NATURE OF HIS OPERATIONS

Heaviside's three volumes on electromagnetic theory are a compilation of papers and articles covering a period from 1891 to 1912, during which time it seems that he was interested in almost every phase of physics and mathematics.

A very large part of this work is devoted to electrical phenomena and, more particularly, to the calculations, by operational methods, of disturbances in electrical networks. Maxwell had already given a most comprehensive theory of almost anything that could happen in electrical systems, but his mathematics, which was conventional mathematics, was applicable only with the greatest difficulty to many practical problems. The task, therefore, that Heaviside set himself was to evolve some new point of view which would simplify the treatment without impairing the accuracy attained. This he succeeded in doing admirably by the introduction of his operational solution and the expansion theorem.

The particular physical problems that he studied led to linear differential equations with constant coefficients and certain types of partial differential equations. In particular, he studied the behavior of systems initially at rest to which forces or impulses were suddenly applied at $t = 0$.

Prior to Heaviside's time, the use of operators was well established. For example, it was customary to write the solution of a differential equation, such as

$$\frac{dy}{dt} - ay = f(t) \quad \text{as} \quad y = \frac{1}{D - a} f(t)$$

where D indicated the operation d/dt

The solution of this equation is, of course,

$$y = C\epsilon^{at} + \epsilon^{at}\int\epsilon^{-at}f(t)dt$$

$C\epsilon^{at}$ being the complementary function and the second term the particular integral. The first term is independent of $f(t)$ and depends upon the terminal conditions only. The second requires integration, often very laborious, especially in the case of differential equations of higher order, such as appear in systems having a large number of degrees of freedom or in systems involving "distributed" constants.

Heaviside found that if the applied force was zero up to time $t = 0$ and then suddenly rose to unity and remained at unity from then on, great simplifications resulted and solutions could be obtained even for the most complicated cases by means of simple algebraic operations.

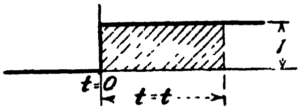

Fig. 1

He called the particular force or e.m.f. discussed above the *unit* force or e.m.f. and designated it by $\mathbf{1}$. Its shape is shown in heavy lines in Fig. 1. Thus, if F represents the force acting on a system, $F\mathbf{1}$ represents a force which is zero up to $t = 0$ and has a constant value F after $t = 0$. Thus, $F\mathbf{1}$ tells a great deal about the nature of the force.

Perhaps it is well to illustrate the distinction between F and $F\mathbf{1}$ by an example, using first the classical method and then the method of Heaviside. If a steady e.m.f. E is applied to an inductive circuit at $t = 0$, the current can be found by solving the following differential equation:

$$E = ir + L\frac{di}{dt} \tag{1}$$

subject to the condition that

$$i = 0 \text{ when } t = 0$$

Incidentally, it requires some understanding of electric circuits to know that the current in this case is zero when $t = 0$.

Heaviside would write

$$E\mathbf{1} = ir + L\frac{di}{dt}$$

In this equation, it is implied that the disturbance in the system begins at $t = 0$ and later is caused by a steady voltage E.

His equation tells the whole story. If properly used, there should be no need for determining the integration constant or for any knowledge about the particular reason why the current must be zero when $t = 0$.

It will be evident later that there is an advantage in introducing the symbol p for the operator d/dt. As a matter of fact, the operator p does all that d/dt can do and more. Introducing p in the equation given above, we get

$$E\mathbf{1} = ir + Lpi = i(r + pL)$$

and solving this we get

$$i = \frac{E}{r + pL}\mathbf{1} \tag{2}$$

Of course, this last equation tells nothing unless we find its interpretation as a function of t.

Heaviside succeeded in doing this, and while he does not state just how it was done, the development was most likely that shown below. Referring to equation (1)

$$\frac{di}{dt} = \frac{1}{L}(E - ir)$$

thus

$$i = \frac{1}{L}\left[\int_0^t E\,dt - r\int_0^t i\,dt\right]$$

If this value of i is substituted under the integral sign, we get as a first approximation

$$i = \frac{E}{L}t - \frac{r}{L}\left[\int_0^t \frac{Et}{L}dt - \frac{r}{L}\int_0^t\int_0^t i\,dt^2\right]$$

Continued substitutions will give

$$i = \frac{E}{L}\left(t - \frac{r}{L}\frac{t^2}{\underline{|2}} + \frac{r^2}{L^2}\frac{t^3}{\underline{|3}} - \cdots + \text{an integral}\right) \tag{3}$$

The power series in t is converging so that in the limit the integral can be neglected.

Now refer to (2) and perform an ordinary division in such a way that the p's appear in the denominators: Then

$$\frac{E\mathit{1}}{r + pL} = \frac{E}{L}\left(\frac{1}{p} - \frac{r}{L}\frac{1}{p^2} + \frac{r^2}{L^2}\frac{1}{p^3} - \cdots\right)\mathit{1} \qquad (4)$$

Comparing (3) and (4), it is seen that they are the same if

$$\frac{1}{p^n}\mathit{1} = \frac{t^n}{\underline{n}} \qquad (5)$$

This relation is given by Heaviside in his "Electromagnetic Theory," Vol. II, page 130.

It may be said that this is well and good for this particular case. But does it work with differential equations of higher order? It can readily be shown that the same relation obtains in all cases. It is always possible to get the solution in t of an operational equation by "algebrizing" it by division and using the relation given in equation (5). The solution is in the form of an infinite series, which, at times, is unfortunate. It will later be shown that by using the expansion theorem another type of solution is obtained

Figure 1 is a chart of the *unit function* $\mathit{1}$. Note that

$\mathit{1} = 0$ when $t < 0$
$\mathit{1} = 1$ when $t > 0$

and

$\mathit{1}$ has all values between zero and 1 at $t = 0$

Let us now differentiate the unit function.

It is seen that

$p\mathit{1} = 0$ when $t < 0$
$p\mathit{1} = 0$ when $t > 0$

and in operational calculus

$p\mathit{1}$ has all values from zero to infinity at $t = 0$.

Before we consider the significance of $\frac{1}{p}\mathit{1}$, it is well to rewrite equation (5) as follows:

$$\frac{1}{p^n}\mathit{1} = \frac{t^n}{\underline{n}}\mathit{1} \qquad (6)$$

This is a more complete statement, although not used explicitly by Heaviside. $t^n/|\underline{n}$ is really multiplied by $\boldsymbol{1}$ because we have assumed that the event begins at $t = 0$ and the $\boldsymbol{1}$ wipes out everything before $t = 0$. For $n = 1$,

$$\frac{1}{p\boldsymbol{1}} = t\boldsymbol{1}$$

Thus, $\frac{1}{p}\boldsymbol{1}$ should be numerically equal to $t\boldsymbol{1}$. This is satisfied if

$$\frac{1}{p}\boldsymbol{1} = \int_0^t \boldsymbol{1}dt \tag{7}$$

where the integral means the shaded area in Fig. 1 and is numerically equal to t. Indeed, we might have written $\frac{1}{p}\boldsymbol{1} = \int_{-\infty}^t \boldsymbol{1}dt$ because nothing is contributed to the integral up to a time ever so near $t = 0$. The relation could also have been written $\frac{1}{p}\boldsymbol{1} = \int_{0-\epsilon}^t \boldsymbol{1}dt$, the lower limit being ever so little to the left of $t = 0$, so as to be sure to include the vertical part of the unit function.

This refinement is, however, not necessary as long as we realize that when we define $\frac{1}{p}\boldsymbol{1}$ as $\int_0^t \boldsymbol{1}dt$, the lower limit is $t = 0 - \epsilon$ as ϵ approaches zero.

In all work involving Heaviside's operators, it should be remembered that we always operate on the unit function and that, as a result, the function of t *which is obtained from the operational solution is actually multiplied by the unit function.*

Under this condition, differentiation and integration are inverse processes. One cancels the other. Thus,

$$p\left(\frac{1}{p}\boldsymbol{1}\right) = \frac{1}{p}(p\boldsymbol{1}) = \boldsymbol{1}$$

The two operations always cancel each other, as will be seen presently. In other words, we shall show that

$$p\left[\frac{1}{p}f(t)\boldsymbol{1}\right] = \frac{d}{dt}\int_0^t f(t)\boldsymbol{1}dt = f(t)\boldsymbol{1} \tag{8}$$

also,

$$\frac{1}{p}[pf(t)\boldsymbol{1}] = \int_0^t \frac{d}{dt} f(t)\boldsymbol{1}\,dt = f(t)\boldsymbol{1} \tag{9}$$

whereas

$$\frac{d}{dt}\int_0^t f(t)dt = f(t)$$

and

$$\int_0^t \frac{d}{dt} f(t)dt = f(t) - f(0)$$

In other words, when the unit function is not involved, the two processes are not always inverse processes. This interesting relation and many other considerations regarding the property of the unit function are dealt with in a paper by Dr. J. J. Smith.[1] For the sake of generality, let the unit function be represented by $H(t)$ in equations (8) and (9). Then (8) becomes

$$\frac{d}{dt}\int_0^t f(t)\,H(t)dt = f(t)H(t) \tag{10}$$

and (9) becomes

$$\int_0^t \frac{d}{dt} f(t)H(t)dt = \int_0^t f'(t)H(t)dt + \int_0^t f(t)H'(t)dt = f(t)H(t)\Big]_0$$
$$- \int_0^t f(t)H'(t)dt + \int_0^t f(t)H'(t)dt = f(t)H(t) - f(0)H(0) \tag{11}$$

If, now, $H(t)$ is the unit function we get

$$\text{equation (10)} = f(t)\boldsymbol{1}$$

and

$$\text{equation (11)} = f(t)\boldsymbol{1} - f(0)\boldsymbol{1}_0$$

The question is, what is the unit function at $t = 0$? It has, according to our definition, all values between zero and unity. We can evaluate equation (11) by changing the lower limit of integration, shifting it ever so little to the left so as to take in the entire unit function. If the lower limit be $t = 0 - \epsilon$, equation (11) becomes $f(t)H(t) - f(0 - \epsilon)H(0 - \epsilon)$. But as the unit function is zero ever so little to the left of the origin, we conclude that $f(0)\boldsymbol{1}_0$ is zero and, hence, that equations (10) and (11) are the same.

[1] *Franklin Inst. Proc.*, October, November, and December, 1925.

If $f(t) = 1$, it follows from equations (8) and (9) that

$$p\left[\frac{1}{p}\mathit{1}\right] = \frac{1}{p}[p\mathit{1}] = \mathit{1}.$$

Incidentally, it is evident from this that $1/p$ is $\int_{0\ \epsilon}^{t}$ as ϵ approaches zero, or, if we prefer, we could also define $1/p$ as $\int_{-\infty}$ and get the same result. We prefer to define it as $\int_{0}^{t}$, since in our problems we are dealing with positive values of t only.

After this discussion of the general properties of the unit function, it may be well to revert to the original problem, which was to find the current in an inductive circuit when an e.m.f. $E\mathit{1}$ is applied.

The operational solution was shown to be

$$i = \frac{E}{r + pL}\mathit{1} = \frac{E}{L}\frac{1}{p + \alpha}\mathit{1},$$

where

$$\alpha = \frac{r}{L}$$

Expanding by the binomial theorem,

$$\begin{aligned} i &= \frac{E}{L}\left(\frac{1}{p} - \frac{\alpha}{p^2} + \frac{\alpha^2}{p^3} - \cdots\right)\mathit{1} \\ &= \frac{E}{L}\left(t - \frac{\alpha t^2}{\underline{|2}} + \frac{\alpha^2 t^3}{\underline{|3}} - \cdots\right)\mathit{1} \\ &= \frac{E}{L\alpha}\left(\alpha t - \frac{\alpha^2 t^2}{\underline{|2}} + \frac{\alpha^3 t^3}{\underline{|3}} - \cdots\right)\mathit{1} \\ &= \frac{E}{r}[1 - \epsilon^{-\alpha t}]\mathit{1}. \end{aligned}$$

We note that the operator

$$\frac{1}{p + \alpha}\mathit{1} = \frac{1}{\alpha}(1 - \epsilon^{-\alpha t})\mathit{1} \tag{12}$$

An operator of much interest is

$$\frac{p}{p + \alpha}\mathit{1} = \left(1 - \frac{\alpha}{p} + \frac{\alpha^2}{p^2} - \cdots\right)\mathit{1} = \left(1 - \alpha t + \frac{\alpha^2 t^2}{\underline{|2}} - \cdots\right)\mathit{1}$$
$$= \epsilon^{-\alpha t}\mathit{1} \tag{13}$$

At this point, one may be and probably is curious to know why in these cases the "algebrizing" by division, as Heaviside calls it, was made in such a way that the p is in the denominator. Why not divide the other way, as follows:

$$\frac{p}{\alpha+p}\mathit{1} = \left(\frac{p}{\alpha} - \frac{p^2}{\alpha^2} + \frac{p^3}{\alpha^3} - \cdots\right)\mathit{1}$$

The answer is that we get no information in this case. Heaviside has given the meaning of $\frac{1}{p^n}\mathit{1}$ as a function of t; therefore, we should make our "algebrizing" such that p appears in the denominator. This matter is discussed further in Chaps. XXIV and XXV.

Returning, for a moment, to equation (13),

Heaviside writes

$$\frac{p}{p+\alpha}\mathit{1} = \epsilon^{-\alpha t}.$$

He omits the unit function on the right-hand side and uses it only, and that not always, in equations involving p but not t.

There is much to be said for omitting the $\mathit{1}$ on the right-hand side. It makes much simpler printing, for one thing, and causes no confusion provided we do not try to operate on the t function by p.

To illustrate the point,

$$p\epsilon^{-\alpha t} = \frac{d}{dt}\epsilon^{-\alpha t} = -\epsilon^{-\alpha t}$$

But $p\epsilon^{-\alpha t}\mathit{1}$ means differentiating a product. Thus,

$$p(f(t)H(t)) = f'(t)H(t) + H'(t)f(t)$$

If

$$H(t) = \mathit{1}$$

Then

$$p(f(t)\mathit{1}) = f'(t)\mathit{1} + f(t)p\mathit{1}$$

But $p\mathit{1}$ is zero for all values of t except at $t = 0$; and at $t = 0$, $f(t) = f(0)$. Hence,

$$p(f(t)\mathit{1}) = \mathit{1}f'(t) + f(0)p\mathit{1} \tag{14}$$

This is an important relation. Thus, if

$$f(t) = \epsilon^{-\alpha t}$$

$$p\epsilon^{-\alpha t}\mathit{1} = -\alpha\epsilon^{-\alpha t}\mathit{1} + \epsilon^{\circ}p\mathit{1} = -\alpha\epsilon^{-\alpha t}\mathit{1} + p\mathit{1}, \text{ or}$$

$p\epsilon^{-\alpha t}\mathit{1}$ is the same as $p\epsilon^{-\alpha t}$ with the addition of an impulse $p\mathit{1}$. This impulse is something that rises to infinity at $t = 0$. Consider again $p\epsilon^{-\alpha t}\mathit{1}$ and write $\epsilon^{-\alpha t}$ in an infinite series. Then,

$$p\epsilon^{-\alpha t}\mathit{1} = p(1 - \alpha t + \frac{\alpha^2t^2}{\underline{|2}} - \frac{\alpha^3t^3}{\underline{|3}} + \cdots)\mathit{1}$$

If we differentiate this as a product, then, from equation (14),

$$p\epsilon^{-\alpha t}\mathit{1} = (-\alpha + \frac{2\alpha^2t}{\underline{|2}} - \frac{3\alpha^3t^2}{\underline{|3}} + \cdots)\mathit{1} + p\mathit{1}$$
$$= -\alpha\epsilon^{-\alpha t}\mathit{1} + p\mathit{1}.$$

The importance of using the $\mathit{1}$ in connection with functions of t when these are operated on by p may be illustrated by another example.

It has been shown that

$$\frac{p}{p + \alpha}\mathit{1} = \epsilon^{-\alpha t}\mathit{1}$$

If we multiply both sides by $p + \alpha$, the result is

$$p\mathit{1} = (p + \alpha)e^{-\alpha t}\mathit{1}$$

Is this equation an identity?

$$(p + \alpha)\epsilon^{-\alpha t}\mathit{1} = -\alpha\epsilon^{-\alpha t}\mathit{1} + \epsilon^{\circ}p\mathit{1} + \alpha\epsilon^{-\alpha t}\mathit{1} = p\mathit{1} \quad \text{Q.E.D}$$

This identity would not have resulted if we had written

$$\frac{p}{p + \alpha}\mathit{1} = \epsilon^{-\alpha t}.$$

Incidentally, it is worth while to call attention to certain other phases which emphasize the importance of retaining the symbol of the unit function.

Some interpreters of Heaviside write

$$\frac{p}{p + \alpha} = \epsilon^{-\alpha t}$$

and, therefore, substituting $-\alpha$ for α,

$$\frac{p}{p - \alpha} = \epsilon^{\alpha t},$$

omitting all unit-function symbols.

From these relations one might well conclude that

$$\frac{p}{p+\alpha}\cdot\frac{p}{p-\alpha}=1.$$

or

$$\frac{p^2}{p^2-\alpha^2}=1$$

As a matter of fact,

$\frac{p^2}{p^2-\alpha^2}\mathit{1}=\cosh \alpha t\mathit{1}$, as can readily be ascertained by "algebrizing." A mistake of this kind is not likely to occur if the equations are given as

$$\frac{p}{p+\alpha}\mathit{1}=\epsilon^{-\alpha t}$$

$$\frac{p}{p-\alpha}\mathit{1}=\epsilon^{\alpha t}$$

provided it is remembered that any $f(t)$ used in the operational calculus is, in reality, multiplied by the unit function. Multiplying these, the student would get a $\mathit{1}^2$ on the left-hand side, and nobody pretends to know the meaning of operators on $\mathit{1}^2$.

Summary.—Linear differential equations with constant coefficients, whose solutions are convergent series in t, can be solved by operational methods if p is substituted for d/dt. The operational equation is subject to the common laws of algebra and is usually brought in the form of an infinite series in terms of $1/p$, $1/p^2$, etc., which are evaluated from the relation

$$\frac{1}{p^n}\mathit{1}=\frac{t^n}{\lfloor n}\mathit{1}.$$

CHAPTER II

THE NATURE AND PRACTICAL VALUE OF THE OPERATIONAL SOLUTION

Heaviside evidently concluded that the use of differentials and integrals was undesirable—if it were possible to do without them. He also believed that even the most primitive electrical engineer could apply Ohm's law to the permanent condition in a direct-current network involving only e.m.fs. and resistances. In short, he adopted the premise that in the circuit illustrated in Fig. 2 any electrical engineer could show that the total resistance is

$$R_0 = \frac{RR_1}{R + R_1}$$

Thus,

$$i = \frac{E}{R_0} = E\frac{(R + R_1)}{RR_1}$$

is the steady current after the system has become stable.

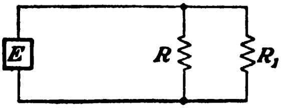

FIG. 2.

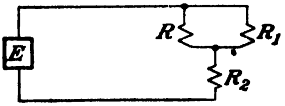

FIG. 3.

Again, in the circuit shown in Fig. 3 he assumed it as evident that the total resistance was

$$R_0 = R_2 + \frac{RR_1}{R + R_1}$$

and that the current in any branch can easily be found as a function of the steady impressed e.m.f.

He then considered more complicated circuits as, for instance, that shown in Fig. 4. Here a resistance is connected in series with an inductance. The instantaneous relation

$$e = ir + L\frac{di}{dt}$$

he assumed as well known. Thus, under his conditions we write

$$E1 = ir + pLi = i(r + pL)$$

In the case shown in Fig. 5, where the circuit consists of a condenser in series with a resistance, the instantaneous relation is

$$e = ir + \frac{1}{C}\int_{0}^{t} idt$$

Thus, under his conditions,

$$E1 = ir + \frac{i}{pC} = i\left(r + \frac{1}{pC}\right)$$

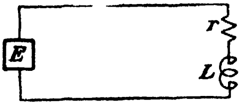

FIG 4.

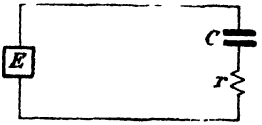

FIG 5.

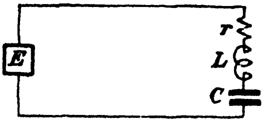

FIG. 6

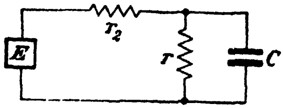

FIG. 7.

In the case shown in Fig. 6,

$$e = ir + L\frac{di}{dt} + \frac{1}{C}\int_{0}^{t} idt$$

Thus, under his conditions,

$$E1 = ir + pLi + \frac{i}{Cp} = i\left(r + pL + \frac{1}{pC}\right)$$

His so-called *resistance operator* for a circuit of resistance and inductance in series is $z = r + pL$. Note that this does not mean that the impedance is $r + pL$. It merely means that if $1/z$ operates on the unit function of E, the current under Heaviside's condition can be found. The resistance operator for a circuit of resistance and capacity in series is $z = r + \frac{1}{pC}$.

The operator in the case of resistance, inductance, and capacity in series is $z = r + pL + \frac{1}{pC}$.

Consider next the circuit shown in Fig. 7, which might represent a leaky condenser in series with a resistance We may proceed to set up the equation of the steady current as if the various branches contained resistances only; *i.e.*, from Fig. 3 we get

$$R_0 = R_2 + \frac{RR_1}{R + R_1}$$

Substituting for the R's the corresponding resistance operators, we get

$$z_0 = r_2 + \frac{r\dfrac{1}{pC}}{r + \dfrac{1}{pC}} = r_2 + \frac{r}{1 + rpC} = \frac{r_2 + rr_2pC + r}{1 + rpC}$$

Therefore,

$$i = \frac{E\mathbf{1}}{z_0} = E\frac{(1 + rpC)}{r + r_2 + rr_2pC}\mathbf{1} \qquad (15)$$

This relation he calls the *operational solution* and writes it

$$i = E\frac{Y}{Z}\mathbf{1} \text{ or } E\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1} \qquad (16)$$

Thus, in this case

$$Y = 1 + pCr$$
$$Z = r + r_2 + rr_2pC$$

It is evident that the operational solution is not limited to finding the current caused by the application of a steady voltage E at $t = 0$. Another problem might be to find, for instance, the operational solution of the e.m.f. consumed by the condenser at any time after the application of E at the terminal of the network. To get this solution, it is at present better for the student to refer to the equivalent direct-current network illustrated in Fig. 3 and to solve for the voltage consumed by R_1, which in that diagram takes the place of the condenser in Fig. 7. This voltage is readily seen to be

$$e = E\frac{RR_1}{RR_2 + RR_1 + R_1R_2}$$

Substituting the resistance operators, we get

$$e = E\frac{r/pC}{rr_2 + \dfrac{r}{pC} + \dfrac{r_2}{pC}}\mathbf{1} = E\frac{r}{rr_2pC + r + r_2}\mathbf{1}$$

Let us return for a moment to equations (15) and (16).

The question may well be asked, why not write the operational solution

$$i = E\frac{\dfrac{1 + pCr}{r + r_2 + rr_2pC}}{1}\mathbf{1}$$

in which case Y would be $\dfrac{1 + pCr}{r + r_2 + rr_2pC}$ and $Z = 1$, or why not write

$$i = E\frac{1}{\dfrac{r + r_2 + rr_2pC}{1 + pCr}}\mathbf{1}$$

in which case Y would be unity and

$$Z = \frac{r + r_2 + rr_2pC}{1 + pCr}$$

It will be clear from what follows in the proof of the expansion theorem that the first relation would not do.

The rule is that the operational solution must be of such form that when Z, the denominator, is equated to zero, we obtain the largest number of roots of p. In the first case, $Z = 0$ gives no roots at all. In most problems, Z contains higher powers of p than the first, when, of course, several roots are obtained.

From this consideration, it should be evident how the operational solution is obtained at least in networks of concentrated resistance, inductance, and capacity.

A number of examples will be given later which involve not only resistance, inductance, and capacity but also mutual inductances. These will help to clear up any doubts that may exist at this time.

CHAPTER III

THE EXPANSION THEOREM

In Chapter II has been shown the method of obtaining the operational solution. It must be remembered that the operational solution gives the answer only in case a steady force is suddenly applied to the system at $t = 0$. This is, of course, not always the case in actual problems where the forces frequently are varying or, indeed, where no external forces are applied, as in the case of the discharge of a condenser. Such cases are easily solved, however, as will be shown in Chaps. VI, VIII, and XIV. As a rule, the solution then depends upon that obtained by first assuming a steady force applied at $t = 0$. Briefly, whatever the nature of the problem, it is usually essential to know the solution in the case of a steady force applied at $t = 0$, that is, under the so-called *Heaviside condition*.

In equation (16) in Chap. II, the operational solution for the line current was given as $i = E\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1}$ where $Y_{(p)}$ and $Z_{(p)}$ were certain functions of p.

If $Z_{(p)}$ is of higher power in p than $Y_{(p)}$, then $Y_{(p)}/Z_{(p)}$ is a fraction of powers. If, on the other hand, $Y_{(p)}$ is of higher power in p than $Z_{(p)}$, it is always possible by division to reduce $Y_{(p)}/Z_{(p)}$ to a sum of terms some of which are integral functions of p and one of them a fraction, involving powers of p in both numerator and denominator. So the resultant expression becomes, for instance, $Ap\mathbf{1} + B\mathbf{1} + C\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1}$. In the last case, the expression $Y_{(p)}/Z_{(p)}$ is obviously not the same as that in the first case. The solution of the problem is, then, the sum of the solutions of $Ap\mathbf{1}$, $B\mathbf{1}$, and $C\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1}$.

From what has been said about $p\mathbf{1}$, it is evident that this term $Ap\mathbf{1}$ contributes something which is infinite at $t = 0$ and zero afterward. The solution of the third term, the fraction in p, is obtained by the expansion theorem.

$Z_{(p)}$ always contains p in various powers. Thus, in general, $Z_{(p)}$ can be written

$$Z_{(p)} = p^n + ap^{n-1} + \cdot \cdot$$

Let the roots corresponding to $Z_{(p)} = 0$ be p_1, p_2, p_3, etc. Therefore,

$$Z_{(p)} = (p - p_1)(p - p_2)(p - p_3) \cdot \cdot \cdot$$

and

$$\frac{Y_{(p)}}{Z_{(p)}} = \frac{Y_{(p)}}{(p - p_1)(p - p_2)(p - p_3) \cdot \cdot \cdot}$$

When this expression is broken into partial fractions and all roots are different and none is zero, we get

$$\frac{Y_{(p)}}{Z_{(p)}} = \frac{A}{p - p_1} + \frac{B}{p - p_2} + \frac{C}{p - p_3} + \cdot \cdot \cdot \qquad (17)$$

in which

$$A = \frac{Y_{(p_1)}}{Z'_{(p_1)}}, \quad B = \frac{Y_{(p_2)}}{Z'_{(p_2)}}, \text{ etc}$$

where $Z'_{(p)}$ is $\frac{d}{dp}Z_{(p)}$ and $Z'_{(p_1)}$ is the particular value of $Z'_{(p)}$ when p_1 is substituted for p. For the explanation of this relation, see, for instance, Todhunter's "Integral Calculus" (second chapter) or Pierce's "Table of Integrals" (p. 14).

We get, also,

$$\frac{Y_{(p)}}{Z_{(p)}}\mathit{1} = A\frac{1}{p - p_1}\mathit{1} + B\frac{1}{p - p_2}\mathit{1} + \cdot \cdot \cdot$$

Substituting the value for $\frac{1}{p - p_1}\mathit{1}$ from equation (12) Chap. I, putting $\alpha = -p_1$, it is easily seen that equation (17) becomes

$$\left[-\frac{A}{p_1} - \frac{B}{p_2} - \cdot \cdot \cdot\right]\mathit{1} + \left[\frac{A\epsilon^{p_1t}}{p_1} + \frac{B\epsilon^{p_2t}}{p_2} + \cdot \cdot \cdot\right]\mathit{1}$$

From equation (17) it is evident that the first parenthesis is $\frac{Y_{(p)}}{Z_{(p)}}\mathit{1}$ for $p = 0$ or, for the sake of brevity, $\frac{Y_{(o)}}{Z_{(o)}}\mathit{1}$. The second parenthesis, after substituting the values of $A, B, \cdot \cdot \cdot$, is

$$\left\{\left[\frac{Y_{(p)}\epsilon^{pt}}{p\frac{dZ}{dp}}\right]_{p=p_1} + \left[\frac{Y_{(p)}\epsilon^{pt}}{p\frac{dZ}{dp}}\right]_{p=p_2} + \cdot \cdot \cdot\right\}\mathit{1}$$

Therefore,

$$i = E\left[\frac{Y_{(o)}}{Z_{(o)}} + \sum_{p_1p_2p_3\cdots} \frac{Y_{(p)}\epsilon^{pt}}{p\,\dfrac{dZ_{(p)}}{dp}}\right]1 \tag{18}$$

which is the same equation as Heaviside gives on page 135 (Vol. II). Note, however, that in this demonstration it has been assumed that all roots of $Z_{(p)} = 0$ are unequal and not zero. The method to be used when several roots are equal, or when one root is zero, will be given in Chap. VII.

As has been already stated Heaviside always omitted writing the unit function 1 in terms involving the independent variable t but usually, though not always, wrote it when the solution was given in some function of p. There is a good practical reason for this—we would otherwise have too many 1's in the equations. For this reason, it seems wise to follow his style, and from now on the unit function will be omitted when the solution is given in t. It should be remembered, however, that it belongs there.

It is an interesting fact that while the expansion theorem is deduced under the assumption that $Y_{(p)}/Z_{(p)}$ is a fraction in powers of p, it is not a necessary limitation under the Heaviside condition. The expansion theorem can be used as given in equation (18) even though the highest power in p is the same in $Y_{(p)}$ and $Z_{(p)}$. Even if the power in p is higher in $Y_{(p)}$ than in $Z_{(p)}$, it is not necessary to divide through. Instead, we can proceed in the usual way with the mental reservation that to the result obtained there must be added a term which is zero before and after $t = 0$ but is infinite at $t = 0$.

Consider the case where the highest power in p is the same in the two expressions:

$$Y_{(p)} = p^n + \alpha p^{n-1} + \beta p^{n-2} + \cdots \gamma$$
$$Z_{(p)} = p^n + ap^{n-1} + bp^{n-2} + \cdots c$$

By division we get

$$\frac{Y_{(p)}}{Z_{(p)}} = 1 + \frac{(\alpha - a)\,p^{n-1} + (\beta - b)\,p^{n-2} \cdots \gamma - c}{p^n + \alpha p^{n-1} + \beta p^{n-2} + \cdots c} = 1 + \frac{Y_{1(p)}}{Z_{(p)}}$$

Here the second term is a fraction in p, and we can without hesitation proceed with the expansion theorem.

The solution of the second term generally contains a constant term $Y_{1(o)}/Z_{(o)}$ and a series of terms involving ϵ^{pt}. The solution of the problem is, then,

$$1 + \frac{Y_{1(o)}}{Z_{(o)}} + \Sigma k\epsilon^{pt}$$

$Y_{1(o)}/Z_{(o)}$ would, of course, be $\frac{\gamma - c}{c}$, so that the total value of the constant term would be $1 + \frac{\gamma - c}{c} = \frac{\gamma}{c}$, which is also the value obtained if we had proceeded without the division. It is also evident that since $Z_{(p)}$ is the same whether we divide or not, the number of roots and the values of the roots corresponding to $Z_{(p)} = 0$ are the same.

In the first case, when we do not divide through, a number of exponential terms result:

$$m_1\epsilon^{p_1t} + m_2\epsilon^{p_2t} + \cdot \cdot \cdot$$

In the second case, the same number of terms result:

$$k_1\epsilon^{p_1t} + k_2\epsilon^{p_2t} + \cdot \cdot \cdot$$

and it happens that under the Heaviside condition $k_1 = m_1$, $k_2 = m_2$, etc.

Since $Y_{1(p)} = Y_{(p)} - p^n - [Z_{(p)} - p^n] = Y_{(p)} - Z_{(p)}$

Therefore, $k_1 = \dfrac{Y_{(p)} - Z_{(p)}}{p\dfrac{dZ}{dp}}$ for $p = p_1$. But this is $\dfrac{Y_{(p)}}{p\dfrac{dZ}{dp}}$ since $Z_{(p)}$ for $p = p_1$ must be zero. Thus, $k_1 = m_1$ and, similarly, $k_2 = m_2$, etc.

Experience has indicated that the student is little likely to go astray in the use of the expansion theorem if he introduces a definite order of procedure.

1. Write down the value of $Z_{(p)}$.
2. Set $Z_{(p)} = 0$ and solve for p, which gives values p_1, p_2, p_3, etc.

If $Z_{(p)}$ is linear in p, only one root p_1 results.

If $Z_{(p)}$ is quadratic in p, two roots p_1 and p_2 result. A cubic equation gives three roots, etc.

3. Differentiate $Z_{(p)}$ with respect to p; *i.e.*, find $dZ_{(p)}/dp$.

4. Write out $p\dfrac{dZ_{(p)}}{dp}$ and $\dfrac{Y_{(p)}}{p\dfrac{dZ_{(p)}}{dp}}$

5. Substitute in step 4, $p = p_1$, $p = p_2 \cdots$, etc.

6. Write $Y_{(p)}/Z_{(p)}$ and substitute $p = 0$ to find $Y_{(0)}/Z_{(0)}$. In general, the solution then contains one constant term $Y_{(0)}/Z_{(0)}$ and a number of terms

$$C_1\epsilon^{p_1t} + C_2\epsilon^{p_2t} + \cdots$$

where C_1 is $\dfrac{Y_{(p)}}{p\dfrac{dZ}{dp}}$ for $p = p_1$, $C_2 = \dfrac{Y_{(p)}}{p\dfrac{dZ}{dp}}$ for $p = p_2$, etc. At this point, the mathematician may say that this is nothing but the usual solution of a linear differential equation where the right-hand member is a constant: and this is, of course, the truth. The beauty of Heaviside's treatment lies in the wonderfully simple method used to determine the integration constants C_1, C_2, etc. They are obtained by the simplest kind of operation involving only the knowledge of differentiation with respect to p.

The simplicity of his method is especially appreciated when dealing with differential equations of infinite order, such as are met in systems of "distributed" constants where the number of integration constants are infinite and often hopelessly difficult to evaluate by conventional methods.

Heaviside really uses two rather distinct methods in solving his problems—one, and the safest one, the expansion theorem; the other, the "algebrizing" method, in which he expands the operator in a power series.

The disadvantage of the first method is that it is often very laborious. $Z_{(p)} = 0$ frequently gives a number of complex roots, difficult to locate.

The disadvantage of the second method is that it leads frequently to series expansions which are very difficult to interpret.

Unfortunately, his operational calculus, which depends upon the expansion of functions in a power series, is poorly fitted to certain problems, as will be shown in Chap. XVI. But there are shorter ways, as will be discussed in that chapter and elsewhere. The use of Duhamel's integral (Chap. XIV) or the vector equations, when dealing with permanent conditions due to periodic forces, is strongly to be recommended. If the "algebrizing" method gives a series which is not convergent or is convergent only over a certain region, do not expect it to tell the whole story; it seldom does. Try then another method. Heaviside says (p. 492, Vol II), in connection with expansions involving divergent series:

"There is more to be said on this subject and I have no doubt a good deal more will be said when proper mathematicians will thoroughly explore divergent series for physical purposes."

Consider, first, the simplest possible case, a storage battery of voltage E connected to a circuit consisting of a non-inductive resistance in series with a resistanceless inductance."

The resistance operator is $r + pL$, and the current under the Heaviside condition

$$i = \frac{E}{r + pL}\mathbf{1} = E\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1}$$

Here $Y_{(p)} = 1$ and $Z_{(p)} = r + pL$

$$Z_{(p)} = 0 \text{ gives one root only, } p_1 = -\frac{r}{L}$$

$$\frac{dZ}{dp} = L,\ p\frac{dZ}{dp} = pL$$

$$\frac{Y_{(p)}}{p\frac{dZ}{dp}} = \frac{1}{pL},\ \left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p = p_1} = \frac{1}{-\frac{r}{L}L} = -\frac{1}{r}$$

$$\frac{Y_{(0)}}{Z_{(0)}} = \frac{1}{r}$$

$$\therefore i = E\left[\frac{1}{r} - \frac{1}{r}\epsilon^{-\frac{r}{L}t}\right] = \frac{E}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

The voltage consumed by the inductance is found in a similar way. It is ipL or

$$e_l = \frac{EpL}{r + pL}\mathbf{1}$$

Here

$$Y_{(p)} = pL,\quad Z_{(p)} = r + pL,\quad p_1 = -\frac{r}{L}$$

and

$$\left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p=p_1} = \frac{-\frac{r}{L}L}{-\frac{r}{L}L} = 1$$

$$\frac{Y_{(0)}}{Z_{(0)}} = 0 \quad \therefore e_r = E\epsilon^{-\frac{r}{L}t}$$

Another simple example is as follows: Find the voltage consumed by the condenser in a circuit consisting of a resistance in series with a condenser. Let the circuit be connected to a storage battery of voltage E.

We have, then, a resistance operator $r + \frac{1}{pC}\cdot$ The current is

$$i = \frac{E}{r + \frac{1}{pC}}\mathbf{1}$$

and the voltage taken by the condenser is $e_c = i\frac{1}{pC}$

$$\therefore e_c = E\frac{\frac{1}{pC}}{r + \frac{1}{pC}}\mathbf{1} = \frac{E}{pCr + 1}\mathbf{1}$$

Here

$$Y_{(p)} = 1 \quad Z_{(p)} = pCr + 1 \quad \therefore p_1 = -\frac{1}{Cr}$$

$$\frac{Y_{(p)}}{p\frac{dZ}{dp}} = \frac{1}{pCr}, \quad \left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p=p_1} = -1$$

$$\frac{Y(0)}{Z(0)} = \frac{1}{1} = 1$$

$$\therefore e_c = E\left[1 - \epsilon^{-\frac{1}{Cr}t}\right]$$

Note that at the moment of closing the switch when $t = 0$, $e_C = 0$; thus, a condenser acts at that moment as a piece of heavy wire—it does not consume any voltage. For $t = \infty$, $e_C = E$.

Thus, if we wait long enough the condenser acts as a wire of infinite resistance.

At the first instant, the condenser acts as if it were short circuited. Later, it acts as an open circuit. An inductance, as is seen by a similar analysis in the first problem, acts just the opposite way.

CHAPTER IV

THE EXPANSION THEOREM APPLIED TO SOME DEFINITE PROBLEMS

To illustrate the application of the expansion theorem to the problem cited in Chap. II when a leaky condenser in series with a resistance is suddenly connected to a storage battery of potential difference E, we have $Z_{(p)} = r + r_2 + rr_2pC$ (see equation (15) and Fig. 7 in Chap. II). Therefore,

$$Z_{(p)} = 0 \text{ gives } p = -\frac{r + r_2}{rr_2C}$$

and

$$p_1 = -\frac{r + r_2}{rr_2C}$$

$$\frac{dZ}{dp} = rr_2C$$

Therefore,

$$p\frac{dZ}{dp} = prr_2C \text{ and } \frac{Y_{(p)}}{p\frac{dZ}{dp}} = \frac{1 + rpC}{prr_2C}$$

$$\left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p=p_1} = \frac{r}{r_2(r + r_2)}$$

$$\frac{Y_{(o)}}{Z_{(o)}} = \frac{1}{r + r_2}$$

Therefore,

$$i = E\left[\frac{1}{r + r_2} + \frac{r}{r_2(r + r_2)}\epsilon^{-\frac{r+r_2}{rr_2C}t}\right]$$

For $t = 0$, $i = \frac{E}{r_2}$ and for $t = \infty$, $i = \frac{E}{r + r_2}$

It is of interest to note that the initial and final currents can always be obtained directly from the operational solution. In this case,

$$i = E\frac{(1 + pCr)}{r + r_2 + pCrr_2}\mathbf{1} \tag{19}$$

When we remember the shape of the unit function, we realize that for $t = 0$, $p = d/dt = \infty$; for $t = \infty$, $p = 0$.

Therefore, for $t = 0$, $p = \infty$, or is very large compared with any finite quantity, and we get

$$i = \frac{EpCr}{pCrr_2} = \frac{E}{r_2}$$[1]

For $t = \infty$, $p = 0$, and $i = E\frac{1}{r + r_2}$.[1]

Other examples involving complex roots will be given in Chaps. VII and VIII.

The operational equation brings out some other interesting features. Since for $t = 0$, $p = \infty$ and for $t = \infty$, $p = 0$, it is evident that if the denominator is of a higher power of p than the numerator, the initial current must be zero.

If numerator and denominator are of the same power in p, the initial current is of a definite value, not zero.

If the numerator is of higher power of p than the denominator, then the initial rush of current is infinite; but after an infinitely short time it decreases to a definite value. It might be well to say here that this condition can never be met in any actual

---

[1] The operational solution is the solution of the permanent alternating-current condition when vector representation is used, if for p is substituted $j\omega$; thus, $p^2 = -\omega^2$, $p^3 = -j\omega^3$, etc.

Heaviside brought out this fact in 1892, or perhaps earlier; it had also been shown by Kennelly in 1892.

The proof is simple when it is recollected that a convenient equation of a revolving vector is

$$I = I\epsilon^{j\omega t}$$

Thus,

$$\frac{d}{dt}I = pI = j\omega I\epsilon^{j\omega t} = j\omega I.$$

Referring then to equation (19),

$$I = E\frac{(1 + j\omega rC)}{r + r_2 + j\omega Crr_2}$$

$$= E\frac{r + r_2 + \frac{r^2}{x_c^2}r_2 + j\frac{r^2}{x_c}}{(r + r_2)^2 + \frac{r^2}{x_c^2}r_2^2}$$

where

$$x_c = \frac{1}{\omega C}$$

problem, whether mechanical or electrical. It would exist in the case, for instance, where a condenser is suddenly connected to a battery with wires of no resistance, which obviously is impossible.

Even then, however, the method throws some light on the subject. We would obviously expect to find an instantaneous infinite rush of current during an exceedingly short time such that $\int i dt$ would be the charge which corresponds to the voltage impressed.

The operational solution obviously is

$$i = E\frac{1}{1/pC}1 = pCE1$$

which means an instantaneous rush of current of infinite magnitude and the charge $= i/p = CE$, which is the proper charge.

Problem.—A condenser made up with two different kinds of dielectric is suddenly connected to a storage battery of voltage E, as in Fig. 8. Find the charge on the middle plate.

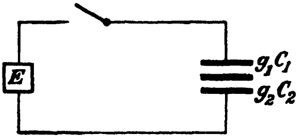

Fig. 8.

Let the leakage conductance and the capacity of the upper portion be g_1 and C_1 and the corresponding values for the lower part be g_2 and C_2. Let e_1 be the instantaneous voltage consumed by the upper part and e_2 that of the lower part, then the current taken by the upper part is obviously $i_1 = e_1(g_1 + pC_1)$ (the resistance operator of the condenser being $1/pC_1$, the corresponding admittance is $\frac{1}{1/pC_1}$). Similarly, the current taken by the lower part is $e_2(g_2 + pC_2)$. These two currents are obviously the same. Thus,

$$e_1(g_1 + pC_1) = e_2(g_2 + pC_2) = i$$

But $e_1 + e_2 = E1$. Therefore,

$$E1 = \frac{i}{g_1 + pC_1} + \frac{i}{g_2 + pC_2}$$

or

$$i = E\frac{(g_1 + pC_1)(g_2 + pC_2)}{g_1 + g_2 + p(C_1 + C_2)}1$$

Therefore,

$$e_1 = \frac{i}{g_1 + pC_1} = E\frac{g_2 + pC_2}{g_1 + g_2 + p(C_1 + C_2)}1$$

and the charge on the top plate is

$$e_1C_1 = EC_1\frac{g_2 + pC_2}{g_1 + g_2 + p(C_1 + C_2)}\mathbf{1}$$

The corresponding charge on the top side of the middle plate is numerically the same but with minus sign.

Consider the lower part of the condenser. Its upper plate has a charge equal to C_2e_2, which is easily shown to be

$$C_2e_2 = EC_2\frac{g_1 + pC_1}{g_1 + g_2 + p(C_1 + C_2)}\mathbf{1}$$

Thus, the total charge on the middle plate is

$$\begin{aligned} q &= E\frac{C_2(g_1 + pC_1) - C_1(g_2 + pC_2)}{g_1 + g_2 + p(C + C_2)}\mathbf{1} \\ &= E\frac{g_1C_2 - g_2C_1}{g_1 + g_2 + p(C_1 + C_2)}\mathbf{1} \\ &= E\frac{A}{G_0 + pC_0}\mathbf{1} \end{aligned}$$

where

$$\begin{aligned} A &= g_1C_2 - g_2C_1 \\ G_0 &= g_1 + g_2 \\ C_0 &= C_1 + C_2 \end{aligned}$$

This can be written

$$q = E\frac{A_1}{p + a}\mathbf{1}$$

where

$$A_1 = \frac{A}{C_0}$$

and

$$a = \frac{G_0}{C_0}$$

The solution can be written down at once by comparing this equation with equation (12) (Chap. I). It is

$$\begin{aligned} q &= E\frac{A}{C_0}\frac{C_0}{G_0}\left[1 - \epsilon^{-\frac{G_0}{C_0}t}\right] \\ &= E\frac{g_1C_2 - g_2C_1}{g_1 + g_2}\left[1 - \epsilon^{-\frac{g_1+g_2}{C_1+C_2}t}\right] \end{aligned}$$

The student should verify this by the use of the expansion theorem.

CHAPTER V

OPERATIONS ON UNIT FUNCTION SQUARED HAVE NO PHYSICAL SIGNIFICANCE

The student may be tempted to write the operational expression for electric power and expect from this to get by the expansion theorem the instantaneous values. He will fail because the operational solution will contain $\mathbf{1}^2$, and neither Heaviside nor anybody else has shown how to operate on $\mathbf{1}^2$.

To illustrate, assume that it is desired to find the power supplied to the inductance of an inductive circuit. The power is the instantaneous product of the voltage consumed by the inductance and the current flowing in the inductance.

The operational solution for the current is

$$i = \frac{E}{r + pL}\mathbf{1}$$

and since the voltage consumed by the inductance is

$$e = pLi = E\frac{pL}{r + pL}\mathbf{1}$$

one might write

$$P = E^2\frac{pL}{(r + pL)^2}\mathbf{1}^2$$

and erroneously proceed to use the expansion theorem. It must be remembered that we know only how to operate on $\mathbf{1}$; we know nothing about rules of operation on $\mathbf{1}^2$.

The solution is the product of the voltage and current when these are given as functions of t, not of p. Thus, the power in this case is

$$P = E^2\frac{\epsilon^{-\frac{r}{L}t}}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

and this is obtained by writing

$$P = \left(\frac{E}{r + pL}\mathbf{1}\right)\left(E\frac{pL}{r + pL}\mathbf{1}\right)$$

and evaluating each term independently.

Incidentally, this difficulty is similar to that encountered when trying to get the power in an alternating-current circuit from the vector expressions of the current and voltage. The student will remember that the answer is obtained by "telescoping" the vectors; ordinary multiplication does not work.

CHAPTER VI

ADDITIONAL OPERATORS EMPLOYED WHEN A NETWORK IS SUDDENLY CONNECTED TO AN ALTERNATOR, INSTEAD OF TO A BATTERY OF CONSTANT VOLTAGE

Referring to equation (13) (Chap. I),

$$\frac{p}{p-a}\boldsymbol{1} = \epsilon^{at}\boldsymbol{1}$$

Thus,

$$\frac{p}{p \pm j\omega}\boldsymbol{1} = \epsilon^{\mp j\omega t}\boldsymbol{1} \text{ when } a = \pm j\omega,$$

and

$$\sin \omega t\boldsymbol{1} = \frac{\epsilon^{j\omega t} - \epsilon^{-j\omega t}}{2j}\boldsymbol{1} = \frac{1}{2j}\left[\frac{p}{p - j\omega}\boldsymbol{1} - \frac{p}{p + j\omega}\boldsymbol{1}\right]$$

$$= \frac{p}{2j}\left[\frac{p + j\omega - p + j\omega}{p^2 + \omega^2}\right]\boldsymbol{1} = \frac{p\omega}{p^2 + \omega^2}\boldsymbol{1}$$

This shows how a sine wave can be converted to an operator on *unit function.*

Similarly

$$\cos \omega t\boldsymbol{1} = \frac{p^2}{p^2 + \omega^2}\boldsymbol{1}$$

Therefore,

$$\sin(\omega t \pm \varphi)\boldsymbol{1} = (\sin \omega t \cos \varphi \pm \cos \omega t \sin \varphi)\boldsymbol{1}$$

$$= \frac{p\omega \cos \varphi \pm p^2 \sin \varphi}{p^2 + \omega^2}\boldsymbol{1}$$

and

$$\cos(\omega t \pm \varphi)\boldsymbol{1} = \frac{p^2 \cos \varphi \mp p\omega \sin \varphi}{p^2 + \omega^2}\boldsymbol{1}$$

This relation, the writer believes, was first shown by Pleijel. Similarly, if "shifting" is used (see Chap. XIII),

$$\epsilon^{-\beta t} \sin(\omega t \pm \varphi)\boldsymbol{1} = \frac{p\omega \cos \varphi \pm p(p + \beta) \sin \varphi}{(p + \beta)^2 + \omega^2}\boldsymbol{1}$$

$$\epsilon^{-\beta t} \cos(\omega t \pm \varphi)\boldsymbol{1} = \frac{p(p + \beta) \cos \varphi \mp p\omega \sin \varphi}{(p + \beta)^2 + \omega^2}\boldsymbol{1}$$

In connection with the last two equations, it should be noted that while $\epsilon^{-\beta t}1 = \frac{p}{p+\beta}1$ and $\sin \omega t1 = \frac{p\omega}{p^2+\omega^2}1$, $\epsilon^{\beta t} \sin \omega t1$ cannot be obtained directly by multiplying the two operators, because the resultant operator would involve the unit function squared.

An application of these operators will be given in the case of an alternator of e.m.f. $E \sin(\omega t + \varphi)$ being suddenly (at $t = 0$) connected to an inductive circuit, as shown in Fig. 9.

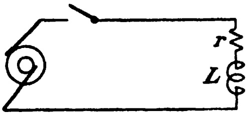

FIG. 9

The procedure is to write first the operational solution as if unit e.m.f. were impressed. It is $i = \frac{E}{r + pL}1$. Then introduce the additional operator which converts the sine wave to a wave of unit function. Thus, the operational solution becomes

$$i = \frac{E}{r + pL} \cdot \frac{\omega p \cos \varphi + p^2 \sin \varphi}{\omega^2 + p^2}1 \tag{20}$$

Therefore,

$$Y_{(p)} = \omega p \cos \varphi + p^2 \sin \varphi$$

and

$$Z_{(p)} = (r + pL)(\omega^2 + p^2)$$

The solution of equation (20) gives the instantaneous values of the current. The denominator is of a higher power of p than the numerator; thus, the initial current value is zero. But in this case we cannot say that for $t = \infty$, $p = 0$. It is not zero, because the final current is obviously not steady; the operational solution gives, of course, the right result. If we desire to know the final value without solving the equation by the expansion theorem, we may resort to the vector representation as discussed previously and write

$$I = \frac{E}{r + pL}$$

where $p = j\omega$.
Therefore,

$$I = \frac{E}{r + j\omega L}$$

which is the well-known solution.

It is now perfectly simple to solve equation (20) by the expansion theorem. Since $Z_{(p)} = (r + pL)(\omega^2 + p^2)$, we get three roots: $p_1 = -\frac{r}{L}$, $p_2 = +j\omega$, $p_3 = -j\omega$. We proceed, then, to find $\frac{dZ_{(p)}}{dp}$ and then $p\frac{dZ_{(p)}}{dp}$, for $p = p_1$, p_2, and p_3.

The result will be three terms ($Y_{(o)}/Z_{(o)}$ is easily seen to be zero in this case), viz ,

$$A\epsilon^{-\frac{r}{L}t} + B\epsilon^{j\omega t} + C\epsilon^{\ j\omega t}$$

The last two terms will combine to a pure sine wave without decrement and are, therefore, the permanent condition, which usually is well known to engineers.

In this case, the permanent alternating current is

$$i_p = \frac{E}{z}\sin(\omega t + \varphi - \alpha)$$

so that it is necessary only to calculate the value of i from the expansion theorem for the root $p = p_1 = -\frac{r}{L}$ and then to add i_p to the solution.

The solution is not always so easily obtained, however. We will, therefore, consider a general simplification of work when the denominator $Z_{(p)}$ is in the form of a product.

Let

$$\frac{Y_{(p)}}{Z_{(p)}} = \frac{Y_{(p)}}{h_{1(p)} \cdot h_{2(p)}}$$

Therefore,

$$Z'_{(p)} = \frac{dZ_{(p)}}{dp} = h_{1(p)}h'_{2(p)} + h_{2(p)}h'_{1(p)}$$

and

$$pZ'_{(p)} = p[h_{1(p)}h'_{2(p)} + h_{2(p)}h'_{1(p)}]$$

When in this expression we substitute the root or roots obtained when $h_{1(p)} = 0$, we get only the second term left, since $h_{1(p)}$

obviously becomes zero. A similar argument applies in regar to the second set of roots, so that the net result is, for instance if $h_{1(p)}$ has only one root, p_1, and $h_{2(p)}$ two roots, p_2 and p_3,

$$i = E\left[\frac{Y_{(0)}}{Z_{(0)}} + \left(\frac{Y_{(p)}\epsilon^{pt}}{ph_{2(p)}h'_{1(p)}}\right)_{p = p_1} + \left(\frac{Y_{(p)}\epsilon^{pt}}{ph_{1(p)}h'_{2(p)}}\right)_{p = p_2} + \left(\frac{Y_{(p)}\epsilon^{pt}}{ph_{1(p)}h'_{2(p)}}\right)_{p = p_3}\right]$$

This relation, the writer believes, was first worked out by Pleije or Herlitz.

In the particular problem under consideration,

$$h_{1(p)} = r + pL;\text{ therefore, } h'_{1(p)} = L \text{ and } ph'_{1(p)} = pL$$
$$h_{2(p)} = \omega^2 + p^2;\text{ therefore, } h'_{2(p)} = 2p \text{ and } ph'_{2(p)} = 2p^2$$

Thus,

$$ph_{2(p)}h'_{1(p)} = pL(\omega^2 + p^2)$$

and

$$ph_{1(p)}h'_{2(p)} = 2p^2(r + pL)$$

$$ph_{2(p)}h'_{1(p)} \text{ for } p = p_1 \text{ becomes } -\frac{r}{L^2}z^2$$

where

$$z^2 = r^2 + \omega^2L^2 = r^2 + x^2$$

$$Y_{(p)} \text{ for } p = p_1 \text{ becomes } z\frac{r}{L^2}\sin(\varphi - a)$$

where

$$\tan\alpha = \frac{x}{r}$$

Thus, the first term becomes

$$-\frac{E}{z}\epsilon^{-\frac{r}{L}t}\sin(\varphi - a)$$

$$ph_{1(p)}h'_{2(p)} \text{ for } p = p_2 = j\omega \text{ becomes } -2\omega^2(r + j\omega L)$$
$$Y_{(p)} \text{ for } p = p_2 \text{ becomes } \omega^2(j\cos\varphi - \sin\varphi)$$

Therefore,

$$\left(\frac{Y_{(p)}}{ph_{1(p)}h'_{2(p)}}\right)_{p=p_2} = \frac{j\cos\varphi - \sin\varphi}{-2(r + j\omega L)}$$

$$= \frac{\cos\varphi + j\sin\varphi}{2j(r + j\omega L)} = \frac{\cos(\varphi - \alpha) + j\sin(\varphi - \alpha)}{2jz}.$$

$$\left(\frac{Y_{(p)}}{ph_{1(p)}h'_{2(p)}}\right)_{p=p_3} = \frac{j\cos\varphi + \sin\varphi}{2(r - j\omega L)}$$

$$= \frac{-\cos(\varphi - \alpha) + j\sin(\varphi - \alpha)}{2jz}.$$

Thus, the last two terms become

$$\frac{E}{z}\left[\cos(\varphi - \alpha)\frac{\epsilon^{j\omega t} - \epsilon^{-j\omega t}}{2j} + \sin(\varphi - \alpha)\frac{\epsilon^{j\omega t} + \epsilon^{-j\omega t}}{2}\right]$$

$$= \frac{E}{z}\sin(\omega t + \varphi - \alpha)$$

which is the permanent value, and the solution is

$$i = \frac{E}{z}\left[\sin(\omega t + \varphi - \alpha) - \epsilon^{-\frac{r}{L}t}\sin(\varphi - \alpha)\right]$$

The process is somewhat lengthy, but as has been stated when it is once done, it is known that the roots $\pm j\omega$ give the permanent condition, so that it is really necessary merely to calculate the condition for the other root or roots, if the permanent condition is known. If not, it is simplest to obtain it first in vector form by substituting $j\omega$ in the operational solution—$i = \frac{E1}{r + pL}$, in this particular case.

This problem will be treated in an entirely different way in Chap. XXVI. The object then will be to bring out certain peculiarities of power-series developments and to show the application of Heaviside's "shifting."

CHAPTER VII

PROCEDURE WHEN THE EXPANSION THEOREM CANNOT BE USED BECAUSE ALL ROOTS ARE NOT DIFFERENT OR SOME ROOTS ARE ZERO

Consider, first, the case when one root is zero. Such a condition would arise if, for instance, it were desired to find the number of coulombs supplied by a storage battery after the switch was closed on an inductive circuit.

The current, then, is $i = \frac{E}{r + pL}\mathbf{1}$

and the number of coulombs is $\int_0^t i\,dt$, or $\frac{i}{p}$

$$\therefore \text{coulombs} = \frac{E}{p(r + pL)}\mathbf{1}$$

In this case, $Z_{(p)} = p(r + pL)$, and one root is zero. Thus, the expansion theorem can not be applied.

The procedure, then, is to determine by the expansion theorem the value of $\frac{E}{r + pL}\mathbf{1}$, which is $\frac{E}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$. Then, since $\frac{1}{p}$ means integration between the limits zero and t, we get

$$\text{coulombs} = \int_0^t \frac{E}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]dt$$

Therefore, when one root is zero, an integration has to be performed.

In general, if $i = E\frac{Y_{(p)}}{p(p - \alpha)(p - \beta)\cdots}\mathbf{1}$, one root is zero. In this case, determine the value of $i_1 = E\frac{Y_{(p)}}{(p - \alpha)(p - \beta)\cdots}\mathbf{1}$ and later find q by integration, that is, $q = \int_0^t i_1 dt.$

When two or more roots are equal, the simplest way is to assume that they differ slightly, in which case the expansion theorem can be applied. The correct answer is later obtained by finding the value of these expressions in t as the roots approach equality or else working out suitable new operators, as will be shown below.

Consider the circuit shown in Fig. 10, consisting of a resistance, inductance, and condenser in series. Then, obviously, the operational solution for the current is

$$i = \frac{E}{r + pL + \frac{1}{pC}}\mathbf{1} \tag{21}$$

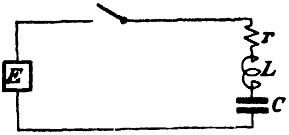

FIG. 10

If the equation is left in its present form,

$$Y_{(p)} = 1$$

$$Z_{(p)} = r + pL + \frac{1}{pC}$$

Thus, $Z_{(p)} = 0$ gives

$$\left.\begin{aligned} p_1 &= -\alpha + j\omega \\ p_2 &= -\alpha - j\omega \end{aligned}\right\} \text{when } \frac{1}{CL} > \frac{r^2}{4L^2}$$

where

$$\alpha = \frac{r}{2L}, \qquad \omega = \sqrt{\frac{1}{CL} - \alpha^2} = \sqrt{\omega_0^2 - \alpha^2}$$

$$\therefore \alpha^2 + \omega^2 = \omega_0^2 \text{ if } \omega_0 = \sqrt{\frac{1}{LC}}$$

$$\frac{dZ_{(p)}}{dp} = L - \frac{1}{p^2C}$$

$$\therefore p\frac{dZ}{dp} = pL - \frac{1}{pC} = \frac{p^2LC - 1}{pC}$$

$$p\frac{dZ}{dp} \text{ for } p = p_1 \text{ then becomes } +2j\omega L$$

and

$$p\frac{dZ}{dp} \text{ for } p = p_2 \text{ becomes } -2j\omega L$$

$$\frac{Y_{(0)}}{Z_{(0)}} = 0$$

$$\therefore i = E\left[\frac{\epsilon^{(-\alpha+j\omega)t}}{2j\omega L} - \frac{\epsilon^{(-\alpha-j\omega)t}}{2j\omega L}\right] = \frac{E}{\omega L}\epsilon^{-\alpha t}\left[\frac{\epsilon^{j\omega t} - \epsilon^{-j\omega t}}{2j}\right]$$

$$= E\frac{\epsilon^{-\alpha t}}{\omega L}\sin \omega t$$

It is readily proved that if $\frac{1}{CL} < \frac{r^2}{4L^2}$,

$$i = E\frac{\epsilon^{-\alpha t}}{L\beta}\sinh \beta t$$

where

$$\beta = \sqrt{\frac{r^2}{4L^2} - \frac{1}{CL}}$$

The case of equal roots, that is, when $\frac{1}{CL} = \frac{r^2}{4L^2}$. As these approach equality, ω approaches zero, and $\sin \omega t$ approaches ωt

$$\therefore i = E\frac{\epsilon^{-\alpha t}}{\omega L}\omega t = Et\frac{\epsilon^{-\alpha t}}{L}$$

But we could have obtained the same relation in another way. Equation (21) might also be written

$$i = \frac{pCE}{rpC + p^2CL + 1}\mathbf{1}$$

or

$$i = \frac{pCE}{CL\left(p^2 + \frac{r}{L}p + \frac{1}{CL}\right)}\mathbf{1} = \frac{pE}{L\left(p^2 + \frac{r}{L}p + \frac{1}{CL}\right)}\mathbf{1}.$$

$$= \frac{pE}{L(p + \alpha)^2}\mathbf{1}$$

The question then, is: What is the meaning of

$$\frac{E}{L}\frac{p}{(p + \alpha)^2}\mathbf{1}?$$

Note that

$$\frac{d}{d\alpha}\frac{p}{p + \alpha} = -\frac{p}{(p + \alpha)^2}$$

but

$$\frac{p}{p+\alpha}1 = \epsilon^{-\alpha t}$$

$$\therefore \frac{d}{d\alpha}\frac{p}{p+\alpha}1 = -t\epsilon^{-\alpha t}$$

$$\therefore \frac{p}{(p+\alpha)^2}1 = t\epsilon^{-\alpha t}$$

Hence

$$i = \frac{E}{L}t\epsilon^{-\alpha t}$$

In the example calculated above, which gave a pair of conjugate roots, the calculation was not very laborious. This was largely due to the fact that $Y_{(p)}$ was unity. This is not always the case.

CHAPTER VIII

SIMPLIFICATION WHICH CAN BE MADE WHEN THE ROOTS ARE CONJUGATES

In general, the operational equation may be written

$$i = E\frac{Y_{(p)}}{Z_{(p)}}1 = E\frac{Y_{(p)}}{p^2 + a_1 p + b}1$$

which, under certain conditions, gives two roots

$$-\alpha \pm j\omega$$

It can easily be proved that when the roots of $Z_{(p)} = 0$ are conjugates, $p\frac{dZ}{dp}$, for $p = p_1$ and for $p = p_2$, must also be conjugate, and so also the two values of $Y_{(p)}$. And, finally, the two expressions of $\frac{Y_{(p)}}{p\frac{dZ}{dp}}$ are conjugates.

Let the complex expression of $\frac{Y_{(p)}}{p\frac{dZ}{dp}}$ for $p = -\alpha + j\omega$ be $\underset{\cdot}{A} = c + jc_1$, or $\underset{\cdot}{A} = A\epsilon^{j\delta}$, where $A = \sqrt{c^2 + c_1^2}$ and $\tan \delta = \frac{c_1}{c}$. Then the corresponding relation for

$$p = p_2 = -\alpha - j\omega \text{ is } \underset{\cdot}{B} = c - jc_1, \text{ or } \underset{\cdot}{B} = A\epsilon^{-j\delta}$$

Therefore, the two terms under the summation sign become

$$\begin{aligned}
&A\epsilon^{j\delta}\epsilon^{(-\alpha+j\omega)t} + A\epsilon^{-j\delta}\epsilon^{(-\alpha-j\omega)t}\\
&= A\epsilon^{-\alpha t}[\epsilon^{j(\omega t+\delta)} + \epsilon^{-j(\omega t+\delta)}]\\
&= 2A\epsilon^{-\alpha t}\cos(\omega t + \delta)
\end{aligned}$$

$$\therefore i = E\left[\frac{Y_{(0)}}{Z_{(0)}} + 2A\epsilon^{-\alpha t}\cos(\omega t + \delta)\right] \tag{22}$$

This method will be demonstrated below.

Find the voltage consumed by the inductance L in Fig. 10 after the switch is closed.

We have

$$i = \frac{E\mathbf{1}}{r + pL + \frac{1}{pC}}$$

$$\therefore e_l = pLi = \frac{pLE}{r + pL + \frac{1}{pC}}\mathbf{1} = \frac{p^2E\mathbf{1}}{p^2 + p\frac{r}{L} + \omega_0^2}$$

$$= \frac{p^2E}{p^2 + 2\alpha p + \omega_0^2}\mathbf{1}$$

$Z = 0$ gives $p = -\alpha \pm j\omega$ where $\omega = \sqrt{\omega_0^2 - \alpha^2}$ and $\omega_0^2 = \dfrac{1}{CL}$

$$\frac{dZ}{dp} = 2(p + \alpha) \therefore p\frac{dZ}{dp} = 2p(p + \alpha)$$

and

$$\frac{Y_{(p)}}{p\dfrac{dZ}{dp}} = \frac{p}{2(p + \alpha)}$$

For $p = -\alpha + j\omega$ this becomes

$$\frac{-\alpha + j\omega}{2(-\alpha + j\omega + \alpha)} = \frac{-\alpha + j\omega}{2j\omega} = \frac{j\alpha + \omega}{2\omega}$$

$$\therefore A = \frac{\sqrt{\alpha^2 + \omega^2}}{2\omega}\underline{/\delta} \text{ where } \tan\delta = \frac{\alpha}{\omega}$$

δ indicating that the vector is offset $\delta°$ or

$$A = \frac{\omega_0}{2\omega}\epsilon^{j\delta}$$

$$\therefore e_l = E\left[\frac{Y_{(0)}}{Z_{(0)}} + \frac{\omega_0}{\omega}\epsilon^{-\alpha t}\cos(\omega t + \delta)\right]$$

but

$$\frac{Y_{(0)}}{Z_{(0)}} = 0$$

$$\therefore e_l = E\frac{\omega_0}{\omega}\epsilon^{-\alpha t}\cos(\omega t + \delta)$$

For $t = 0$,

$$e_l = E\frac{\omega_0}{\omega}\cos\delta = E\frac{\omega_0}{\omega}\frac{\omega}{\omega_0} = E$$

CHAPTER IX

PROBLEM INVOLVING MUTUAL INDUCTANCE

Figure 11 is a diagram of two circuits having mutual inductance for which the conventional relation is

$$e_1 = i_1 r_1 + L_1\frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + M\frac{di_2}{dt}$$

$$0 = i_2 r_2 + L_2\frac{di_2}{dt} + \frac{1}{C_2}\int i_2 dt + M\frac{di_1}{dt}$$

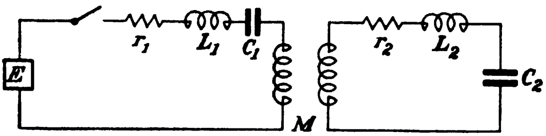

FIG. 11.

Thus, under Heaviside conditions,

$$E\mathbf{1} = i_1\left(r_1 + pL_1 + \frac{1}{pC_1}\right) + i_2 pM = i_1 a_1 + i_2 b$$

$$0 = i_2\left(r_2 + pL_2 + \frac{1}{pC_2}\right) + i_1 pM = i_2 a_2 + i_1 b$$

From the latter equation,

$$i_2 = -\frac{i_1 b}{a_2}$$

$$\therefore E\mathbf{1} = i_1 a_1 - i_1\frac{b^2}{a_2} = i_1\frac{(a_1 a_2 - b^2)}{a_2}$$

$$\therefore i_1 = \frac{Ea_2}{a_1 a_2 - b^2}\mathbf{1}$$

and

$$i_2 = -\frac{Eb}{a_1 a_2 - b^2}\mathbf{1}$$

In these equations,

$$a_1 = r_1 + pL_1 + \frac{1}{pC_1}$$

$$a_2 = r_2 + pL_2 + \frac{1}{pC_2}$$

$$b = pM$$

Special Case.—The condensers are omitted in each circuit.

$E = 500$ volts $\quad r_1 = 100$ ohms $\quad r_2 = 20$ ohms

$L_1 = 0.4$ henry $\quad L_2 = 0.1$ henry

$M = 0.6\sqrt{L_1L_2} = 0.12$ henry

Verify that the operational solution for the primary current is

$$i_1 = 500\frac{20 + 0.1p}{2{,}000 + 18p + 0.0256p^2}\mathbf{1}$$

and that when the expansion theorem has been applied

$$i_1 = 5[1 - 0.412\epsilon^{-138t} - 0.6\epsilon^{-565t}]$$

The secondary current is $i_2 = -5.5\,[\epsilon^{-138t} - \epsilon^{-565t}]$

Problem.—If, instead of applying a storage battery of 500 volts to the primary circuit, a 60-cycle alternator with the same maximum voltage were used, and if it so happened that the switch were closed at the instant when the e.m.f. wave passed through its zero value, the voltage wave would be 500 sin 377*t*.

The operational equation for the primary current would then be

$$i_1 = 500\frac{377p}{p^2 + 377^2} \times \frac{20 + 0.1p}{2{,}000 + 18p + 0.0256p^2}\mathbf{1}.$$

and the numerical value after the expansion theorem is applied would be

$$i_1 = 3.06 \sin (\omega t - 41.3^\circ) + 0.66\,\epsilon^{-138t} + 1.365\epsilon^{-565t}$$

CHAPTER X

THE EFFECT OF SUDDENLY CHANGING THE E.M.F. IN A SYSTEM OR OF SHORT-CIRCUITING A NETWORK

Originally, switch s_1 in Fig. 12 is open and s_2 closed At $t = 0$, switch s_1 is closed, when a current

$$i_1 = \frac{E_1}{r + pL}1 = \frac{E_1}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

flows in the circuit. At $t = t_1$, switch s_2 is opened, which introduces an additional voltage E_2. This voltage causes a current

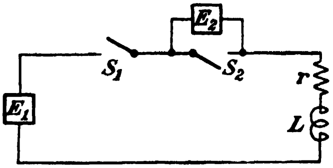

FIG 12

$$i_2 = \frac{E_2}{r + pL}1 = \frac{E_2}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

to flow. which is superimposed on i_1 and begins, of course, at $t = t_1$.

If E_2 is equal but opposite to E_1, then i_2 is negative, as shown by the dotted line in Fig. 13, and the resultant current is shown by the heavy line.

Up to time t_1 the current is $\frac{E_1}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$; after that, it is

$$\frac{E}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right] - \frac{E}{r}\left[1 - \epsilon^{-\frac{r}{L}(t-t_1)}\right].$$

Obviously, after $t = t_1$ there is no battery e.m.f. in the circuit, so that the circuit is short circuited upon itself.

Special Case.—If the short circuit takes place some time after the battery has been connected, so that the system has reached the steady state, the current due to the original e.m.f. is $E\frac{Y_{(0)}}{Z_{(0)}}$, and the current due to the fictitious negative voltage is

$$-E\left[\frac{Y_{(0)}}{Z_{(0)}} + \Sigma\frac{Y_{(p)}}{p\frac{dZ_{(p)}}{dp}}\epsilon^{pt}\right]$$

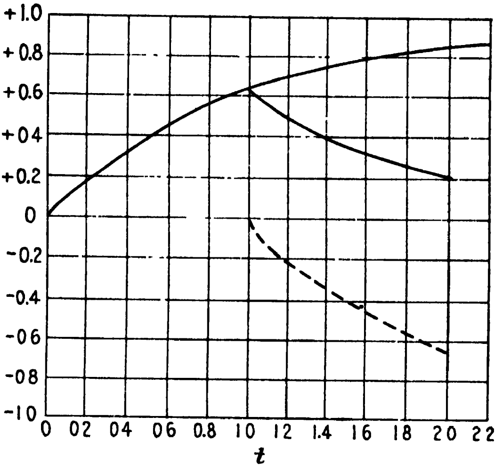

FIG. 13.

where t is counted from the time of short circuit. Therefore, the current after the short circuit is the sum of these two or

$$i = -\Sigma\frac{Y_{(p)}\epsilon^{pt}}{p\frac{dZ_{(p)}}{dp}}$$

The rule is, therefore: Write out the operational solution covering the initial condition (the starting condition), solve this equation by the expansion theorem, omit the constant term, and reverse the sign before the summation term.

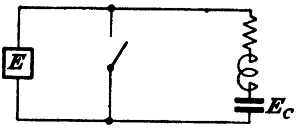

FIG 14.

In this particular circuit, it is easily seen that after the short circuit, $i = \frac{E}{r}\epsilon^{-\frac{r}{L}t}$, but $\frac{E}{r}$ is the current that existed when the short circuit took place. Thus, the original current dies down according to the exponential law.

Another Example.—Find the discharge current of a condenser initially charged to a potential difference E_C from a battery of voltage E in the circuit represented in Fig. 14.

$$i = -\frac{E\mathbf{1}}{r + pL + \dfrac{1}{pC}} = \frac{E_c\mathbf{1}}{r + pL + \dfrac{1}{pC}} \tag{23}$$

Find the voltage across the condenser after the discharge has begun.

$$e_c = \frac{i}{pC} = \frac{E_c}{pC\left(r + pL + \dfrac{1}{pC}\right)}\mathbf{1}$$

where the transient term alone is used after the solution from the expansion theorem has been found.

At first glance, equation (23) seems strange. It contains $E_C\mathbf{1}$, yet we know that the voltage of the condenser is going to decrease probably through oscillations until it becomes zero. It is far from a unit function.

It must be remembered that equation (23) was not derived under the assumption of a steady condenser voltage. It happened to come out that way through mathematical reasoning.

It is interesting to note that in the case of discharge of a condenser, the current is identical with the original charging current, except for reversed direction.

Other Examples.—The oscillations in mechanical systems are similar to those in electric circuits. When dealing with springs of negligible mass compared with the mass of the moving body, and when the frictional force is proportional to the velocity, the equations become identical.

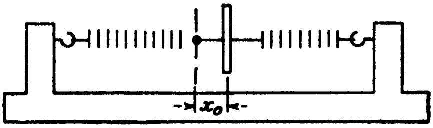

Fig. 15.

The dotted line in Fig. 15 is the position of equilibrium. A force has been applied to the mass so that it has been displaced a distance x_0. Find the position of the mass when the force is suddenly removed.

Let the friction force be $fv = f\dfrac{dx}{dt}$

Let the force necessary to compress $= kx$.

The force necessary to accelerate the mass is $= M\dfrac{d^2x}{dt^2}$.

Then, if a steady force F is applied to bring the mass to position x_0, the following relation exists:

$$F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx$$

or with Heaviside's notation,

$$F1 = Mp^2x + fpx + kx = x(p^2M + pf + k)$$

$$\therefore x = \frac{F}{p^2M + pf + k}1 \tag{24}$$

If the numerical relations among M, f, and k are suitable, the mass will reach its final position after a number of oscillations, and the final value $x_0 = Y_{(0)}/Z_{(0)}$, which corresponds to $p = 0$, will be F/k.

After the force is removed, the motion will be the resultant of two motions: one the original due to $+F1$ and the other due to $-F1$. Thus,

$$x = \frac{Y_{(0)}}{Z_{(0)}} - \left[\frac{Y_{(0)}}{Z_{(0)}} + \Sigma\frac{Y_{(p)}\epsilon^{pt}}{p\dfrac{dZ}{dp}}\right] = -\Sigma\frac{Y_{(p)}}{p\dfrac{dZ}{dp}}\epsilon^{pt} \tag{25}$$

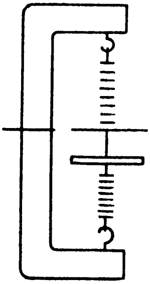

FIG. 16.

Equation (24) may be written

$$x = \frac{F/M}{p^2 + p\dfrac{f}{M} + \dfrac{k}{M}}1$$

$$Z_{(p)} = 0 \text{ gives } p = -\frac{f}{2M} \pm \sqrt{\frac{f^2}{4M^2} - \frac{k}{M}}$$

If the constants are such that $\dfrac{k}{M} > \dfrac{f^2}{4M^2}$ then

$$p = -\alpha \pm j\omega$$

where

$$\alpha = \frac{f}{2M}$$

$$\omega = \sqrt{\omega_0^2 - \alpha^2} \text{ and } \omega_0^2 = \frac{k}{M}$$

$$\frac{dZ}{dp} = 2p + 2\alpha = 2(p + \alpha)$$

thus,

$$p\frac{dZ}{dp} = 2p(p + \alpha) \quad Y_{(p)} = \frac{F}{M}$$

$$\therefore \frac{Y_{(p)}}{p\frac{dZ}{dp}} = \frac{F}{2pM(p + \alpha)}$$

$$\left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p = -\alpha + j\omega} = -\frac{F}{2M\omega(\omega + j\alpha)}$$

$$= -\frac{F(\omega - j\alpha)}{2M\omega(\omega^2 + \alpha^2)} = \frac{F(-\omega + j\alpha)}{2M\omega\omega_0^2}$$

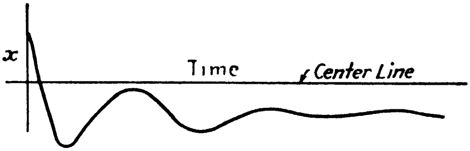

FIG. 17.

Thus referring to Chap. VIII,

$$A = \frac{F}{2M\omega\omega_0} \text{ and } \tan\delta = \frac{+\alpha}{-\omega}$$

δ, therefore, lies in the second quadrant and

$$x = -\frac{F}{M\omega\omega_0}\epsilon^{-\alpha t}\cos(\omega t + \delta)$$

At $t = 0$, $x = -\frac{F}{M\omega\omega_0}\cos\delta$

$$= -\frac{F}{M\omega\omega_0}\left(-\frac{\omega}{\omega_0}\right) = \frac{F}{M}\frac{M}{k} = \frac{F}{k} = x_0$$

At $t = \infty$, $x = 0$. The oscillation begins at $x = x_0$ and ends in the middle position.

If the springs are vertical, as shown in Fig. 16, the equilibrium position will be below the center line due to the weight of the disk.

Problem.—Suppose that a force F greater than the weight had originally been applied so that the position of the disk was that indicated above the dotted line and that the force F had been suddenly removed. Give the equation of x_1, the distance of the disk from the center line after the force was removed. In this case, the following relation holds during the first part of the cycle, that is, during the time that the mass is being raised

$$F - W = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx$$

$F - W$ being the force available for accelerating the mass, overcoming the resistance, and compressing the spring.

$$\therefore (F - W)\mathbf{1} = x(p^2M + fp + k)$$

or

$$x = \frac{(F - W)}{p^2M + fp + k}\mathbf{1} \tag{26}$$

The final position

$$x_0 = \frac{F - W}{k}$$

During the second part of the cycle, that is, when the external force has been removed and the system is free to oscillate under the action of the force of gravity alone, we can apply the same reasoning, or, briefly, we find a new fictitious force F' and calculate x_1 due to that force, then add to this the value of x_0 to get the answer. F' must be such that when added to the original force $F - W$ the result will be $-W$.

$$\therefore F' + F - W = -W$$

or

$$F' = -F$$

therefore,

$$x_1 = -\frac{F}{p^2M + pf + k}\mathbf{1}$$

The solution of this equation, using the expansion theorem, is the sum of a constant term and an exponential term. It is

$$x_1 = -\frac{F}{k} - \frac{F\epsilon^{-\alpha t}}{M\omega\omega_0}\cos(\omega t + \delta)$$

$$\therefore x = x_0 + x_1 = \frac{F - W}{k} - \frac{F}{k} - \frac{F\epsilon^{-\alpha t}\cos(\omega t + \delta)}{M\omega\omega_0}$$

$$= -\frac{W}{k} - \frac{F\epsilon^{-\alpha t}\cos(\omega t + \delta)}{M\omega\omega_0}$$

For

$$t = 0, \quad x = -\frac{W}{k} + \frac{F}{M\omega_0^2} = \frac{F - W}{k}$$

For

$$t = \infty, \quad x = -\frac{W}{k}$$

The chart would be similar to that shown in Fig. 17, if $F = 2W$.

CHAPTER XI

THE EFFECT OF SHORT-CIRCUITING CERTAIN SECTIONS OF A NETWORK AND OF INTRODUCING NEW IMPEDANCES BY CLOSING A SWITCH

Consider as an illustration the circuit in Fig. 18. Switch S is closed after the original current is steady; find the equation of the current after the switch is closed.

Before the switch is closed, a steady current $I = \frac{E}{r + r_1}$ flows in the circuit and an e.m.f. $e = Ir_1$ is consumed by resistance r_1.

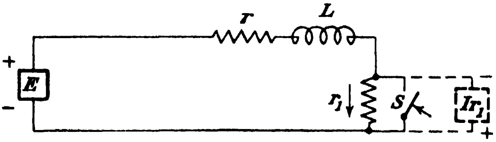

FIG. 18.

The e.m.f. itself is obviously $-Ir_1$ and this e.m.f. exists across the switch blade. After the switch is closed, there is no e.m.f. across the switch blade, so that closing the switch is equivalent to adding suddenly at the switch an e.m.f. $+Ir_1$, shown in dotted lines. The sum of the e.m.f's across the switch blade is, therefore, zero.

This e.m.f. $+Ir_1\mathbf{1}$ causes a certain current to flow in the circuit so that the actual current after the switch is closed is the algebraic sum of the original current and the transient current due to an e.m.f. $Ir_1\mathbf{1}$ at the switch

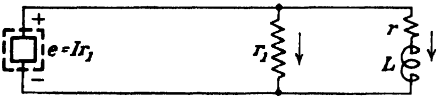

FIG. 19.

It is advantageous to redraw the diagram as shown in Fig. 19 (the resistance of the original battery is assumed as zero).

The transient current in circuit $r + pL$ is

$$i = \frac{Ir_1}{r + pL}1 = \frac{Ir_1}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

The original current was I and was in the same direction as the transient. Thus, the total current is

$$i = I + I\frac{r_1}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

At the moment that the switch closes, $t = 0$, the current begins at its original value and then rises to a final value E/r at $t = \infty$.

In the other branch, the transient current due to e.m.f. Ir_1 rises at once to $I\frac{r_1}{r_1} = I$, and this current flows, as seen in the diagram, in opposite direction to that originally in the resistance. Therefore, the current in r_1 falls at once to zero after the switch is closed.

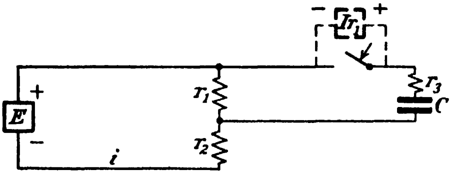

Fig. 20

Figure 20 illustrates a similar case where in closing a switch the constants of the circuit are changed. The method of solving the problem is practically the same as in the previous case. A fictitious e.m.f. is introduced between the switch blades, the value of which is the e.m.f. consumed in the shunted branch just prior to closing the switch. This fictitious e.m.f. causes currents to flow in the network, and these currents added to the original currents give the actual current.

Figure 21 is Fig. 20 redrawn for convenience of analysis.

Problem.—Find current i after the switch is closed. Comparing the diagrams, it is readily seen that current i is the current in resistance r_2.

The resistance operator for the entire circuit (Fig. 21) is

$$z = r_3 + \frac{1}{pC} + \frac{r_1 r_2}{r_1 + r_2}$$

the total transient current, $i_0 = \frac{I r_1}{z}\mathbf{1}$

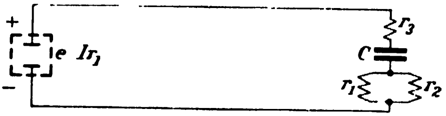

FIG. 21.

This current divides through the two resistances so that

$$i_2 = \frac{r_1}{r_1 + r_2} i_0$$

$$\therefore i_2 = \frac{I r_1^2}{r_1 + r_2} \frac{pC(r_1 + r_2)}{pCr_3(r_1 + r_2) + r_1 + r_2 + pCr_1r_2}\mathbf{1}$$

$$= \frac{I r_1^2 pC}{r_1 + r_2 + pC\,(r_1r_2 + r_1r_3 + r_2r_3)}\mathbf{1}$$

and the actual current i is the sum of the original and this transient. That is,

$$i = \frac{E}{r_1 + r_2}\left[1 + \frac{pCr_1^2}{r_1 + r_2 + pC(r_1r_2 + r_1r_3 + r_2r_3)}\right]\mathbf{1}$$

The first term is simply $\frac{E}{r_1 + r_2}$. The second may be solved either from the operator directly or by the expansion theorem. It may be written

$$\frac{Ap}{B + Dp}\mathbf{1} = \frac{A}{D}\frac{p}{p + \alpha}\mathbf{1}$$

where

$$\alpha = \frac{B}{D}$$

or

$$\frac{Ap}{B + Dp}$$

$$= \frac{A}{D}\epsilon^{-\alpha t} = \frac{Cr_1^2}{C(r_1r_2 + r_1r_3 + r_2r_3)}\epsilon^{-\alpha t}$$

$$\therefore i = \frac{E}{r_1 + r_2}\left[1 + \frac{r_1^2}{r_1r_2 + r_1r_3 + r_2r_3}\epsilon^{-\frac{r_1 + r_2}{r_1r_2 + r_1r_3 + r_2r_3}t}\right]$$

For

$$t = 0$$

$$i = \frac{E(r_1 + r_3)}{r_1 r_2 + r_1 r_3 + r_2 r_3}$$

For

$$t = \infty$$

$$i = \frac{E}{r_1 + r_2}$$

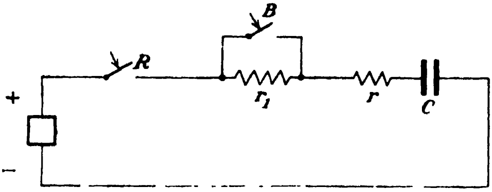

FIG. 21a

While this chapter deals primarily with the effect of switching after the circuit has reached its stable state, the same reasoning applies even though that be not the case.

Consider the case shown in Fig. 21a, which is redrawn in Fig. 21b. Switch R is closed at $t = 0$ and switch B at $t = t_1$. The problem is to find the current at any time.

In Fig. 21b a battery has been substituted which gives a peculiar e.m.f., namely, that which would exist across r, if the circuit were left in its original state. This battery causes a current which adds itself to the original current in the condenser but subtracts itself from that originally in resistance r_1. Since the condenser carries the whole current before as well as after switch B is closed, it is evidently the current that is desired. The original current through the battery is

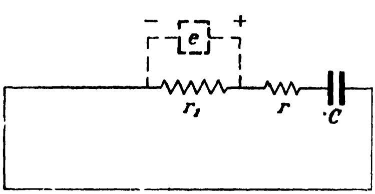

FIG 21b

$$i_1 = \frac{E}{r + r_1 + \frac{1}{pC}}\mathbf{1} = \frac{E}{r_0}\epsilon^{-\alpha_0 t}$$

where

$$r_0 = r + r_1 \text{ and } \alpha_0 = \frac{1}{Cr_0}$$

The voltage consumed by the resistance r_1, which is shown as e, is obviously $e = i_1 r_1 = \frac{Er_1}{r_0}\epsilon^{-\alpha_0 t}$.

This voltage causes a current to flow in the circuit $r + \frac{1}{pC}$, which is

$$i_2 = \frac{E\frac{r_1}{r_0}}{r + \frac{1}{pC}}(\epsilon^{-\alpha_0 t}\mathit{1})$$

$$= E\frac{r_1}{r_0}\frac{pC}{rpC + 1}(\epsilon^{-\alpha_0 t}\mathit{1}) = E\frac{r_1}{rr_0}\frac{p}{p + \alpha}(\epsilon^{-\alpha_0 t}\mathit{1})$$

where

$$\alpha = \frac{1}{Cr}$$

It will be shown in Chap XIII how $\epsilon^{-\alpha_0 t}$ is "shifted" to the left and we get

$$i_2 = \epsilon^{-\alpha_0 t}\frac{Er_1}{rr_0}\frac{p - \alpha_0}{p - \alpha_0 + \alpha}\mathit{1}$$

$\frac{p - \alpha_0}{p + \alpha - \alpha_0}\mathit{1}$ can be solved by the expansion theorem and becomes

$$\frac{1}{\alpha - \alpha_0}\left[-\alpha_0 + \alpha\epsilon^{-(\alpha - \alpha_0)(t - t_1)}\right]$$

$t - t_1$ is used, since time begins after switch B is closed.

$$\therefore i_2 = \epsilon^{-\alpha_0 t}E\frac{r_1}{rr_0}\left[\frac{\alpha}{\alpha - \alpha_0}\epsilon^{-(\alpha - \alpha_0)(t - t_1)} - \frac{\alpha_0}{\alpha - \alpha_0}\right]$$

$$= E\epsilon^{-\alpha_0 t}\left[\frac{1}{r}\epsilon^{-(\alpha - \alpha_0)(t - t_1)} - \frac{1}{r_0}\right]$$

Is this what could be expected? At $t = t_1$, the new current is $\frac{Er_1\epsilon^{-\alpha_0 t_1}}{rr_0}$. We know that at a sudden change of impressed e.m.f. on a condenser it acts as a copper wire; thus, the instantaneous current due to e.m.f. $E\frac{r_1}{r_0}\epsilon^{-\alpha_0 t}$, which is the e.m.f. that exists at $t = t_1$, is merely this e.m.f. divided by the resistance r in series with the

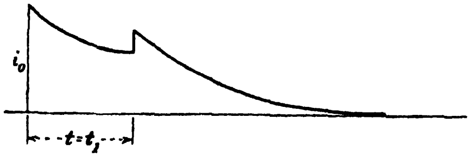

FIG. 21c.

condenser, which checks the result. The total current flowing after $t = t_1$ is $i_0 = i_1 + i_2$ or

$$i_0 = E\epsilon^{-\alpha_0 t}\left[\frac{1}{r_0} - \frac{1}{r_0} + \frac{1}{r}\epsilon^{-(\alpha-\alpha_0)(t-t_1)}\right]$$

$$= E\frac{\epsilon^{-\alpha_0 t}}{r}\epsilon^{-(\alpha-\alpha_0)(t-t_1)}$$

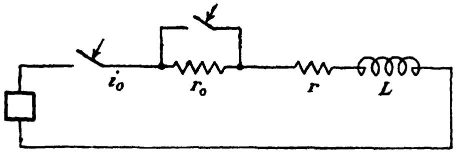

FIG 21d

Up to time $t = t_1$, the current is

$$i_0 = \frac{E}{r_0}\epsilon^{-\alpha_0 t}$$

The chart looks something like that shown in Fig. 21c. The student should work out a similar problem, where instead of

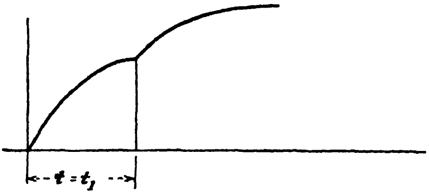

FIG. 21e.

the condenser, a coil of inductance L is used, as shown in Fig. 21d. The solution is, up to time $t = t_1$,

$$i_0 = \frac{E}{r_0}\left[1 - \epsilon^{-\frac{r_0}{L}t}\right]$$

after time t_1 it is

$$i_0 = \frac{E}{r} - \frac{Er_1}{rr_0}\epsilon^{-\frac{r}{L}(t-t_1)} - \frac{E}{r_0}\epsilon^{-\frac{r_1}{L_1}t_1}\epsilon^{-\frac{r}{L}t}$$

and the chart is something like that shown in Fig. 21e.

Switching Problem Involving Alternating-current Voltage. – Referring to Fig. 21f, which is redrawn in Fig. 21g, the switch across r_1 is closed after the circuit is in steady alternating-current state at an instant when the e.m.f. wave passes through its zero value.

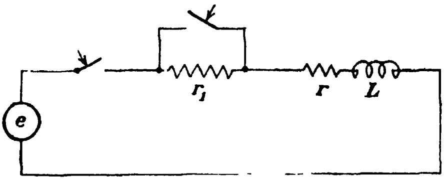

FIG. 21f

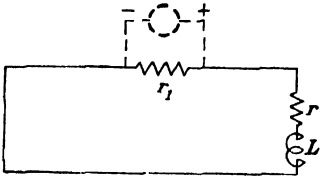

FIG. 21g.

Give the equation of the current before and after the switch is closed.

Before the switch is closed,

$$i = \frac{e}{r + r_1 + pL}$$

Using vector representation,

$$I = \frac{E}{r + r_1 + j\omega L} = \frac{e}{Z_0}\underline{\left|-\varphi\right.}$$

where

$$Z_0 = \sqrt{(r + r_1)^2 + \omega^2 L^2}$$

and

$$\tan\varphi = \frac{\omega L}{r + r_1}$$

Thus, if

$$e = E\sin\omega t, \qquad i = \frac{E}{Z_0}\sin(\omega t - \varphi)$$

and the voltage consumed by the resistance r_1 is $e_1 = \frac{Er_1}{Z_0}\sin(\omega t - \varphi)$.

Now we replace the switch by an alternator of voltage e_1 as shown in Fig. 21g. This alternator sends a current i_2

through $r + pL$ which is in the same direction as the original current i_1, and this current is $\frac{1}{r + pL}e_1$. This current has already been studied in Chap. VI.

It is

$$i_2 = \frac{e_1}{Z}\left[\sin(\omega t - \varphi - \alpha) - \epsilon^{-\frac{r}{L}t}\sin(-\varphi - \alpha)\right]$$

$$= \frac{Er_1}{Z_0 Z}\left[\sin(\omega t - \varphi - \alpha) + \epsilon^{-\frac{r}{L}t}\sin(\varphi + \alpha)\right]$$

where

$$Z = \sqrt{r^2 + \omega^2 L^2}$$

and

$$\tan\alpha = \frac{\omega L}{r}$$

$$\therefore i_0 = \text{total current after the switch is closed}$$

$$= \frac{E}{Z_0}\left[\sin(\omega t - \varphi) + \frac{r_1}{Z}\sin(\omega t - \varphi - \alpha) + \frac{r_1}{Z}\epsilon^{-\frac{r}{L}t}\sin(\varphi + \alpha)\right]$$

Check as to whether or not this is reasonable. For $t = 0$, the two last terms cancel out, and we get the right result. For $t = \infty$, the third term disappears, and when the remaining two are combined and simplified, we get $i = \frac{E}{Z}\sin(\omega t - \alpha)$, which is the right result.

CHAPTER XII

THE EFFECT OF SUDDENLY CHANGING THE CIRCUIT CONSTANTS BY OPENING A SWITCH

Referring to Fig. 22, assume that the circuit has reached its permanent condition before switch S is opened. Before it is opened, a steady current I flows through. After it is opened, zero current flows through it.

If, therefore, we substitute for the mechanical opening of the switch a certain peculiar e.m.f. in the switch, which at all times after $t = 0$ gives a steady current $-I$, we have accomplished the same result Figure 22 might advantageously be redrawn, as shown in Fig. 23

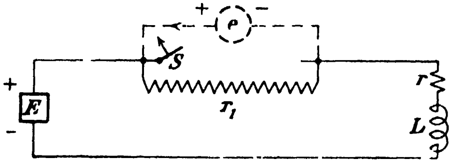

FIG. 22.

The current caused by the peculiar e.m.f. divides in the two circuits in parallel, so that the current i_1 in r_1 and the total current I are related as the admittances.

$$\frac{i_1}{I} = \frac{1/r_1}{\dfrac{1}{r_1} + \dfrac{1}{r + pL}}$$

$$\therefore i_1 = \frac{r + pL}{r + r_1 + pL} I\mathbf{1}$$

Note that unit function $\mathbf{1}$ is inserted. This is evident because the original current in the switch was steady and was $+I$. In order that the current shall be zero, $-I$ must have unit shape.

In this particular case, the initial current $I = E/r$, so that the current flowing in the circuit is

$$i_1 = \frac{E}{r}\left(\frac{r + pL}{r + r_1 + pL}\right)\mathbf{1}$$

For $t = 0$,

$$i_1 = \frac{E}{r}$$

The solution of the equation is easily found to be

$$i_1 = \frac{I}{r + r_1}\left[r + r_1 \epsilon^{-\frac{r + r_1}{L}t}\right]$$

This is the total current flowing, because before the switch was opened there was no current in r_1.

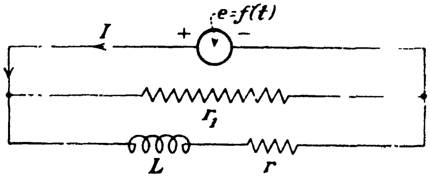

FIG. 23

If the switch had been shunted by a condenser, as in Fig. 24, the following relation would at once be apparent:

$$\frac{i_c}{I} = \frac{pC}{pC + \frac{1}{r + pL}}$$

$$\therefore i_c = \frac{E}{r}\left(\frac{pC(r + pL)}{rpC + p^2CL + 1}\right)1 \qquad (27)$$

An interesting problem is the so-called *buzzer excitation* used with wave meters in a fundamental circuit such as is shown in Fig. 25. Here the equation of the current after the magnet has opened the circuit is obviously the same as in the above case.

Solution of Equation (27) by the Expansion Theorem.—The expression is really the sum of the two following terms:

$$\frac{pA_1}{p^2 + 2\alpha p + \omega_0^2}1 \text{ and } \frac{p^2B_1}{p^2 + 2\alpha p + \omega_0^2}1$$

in which

$$A_1 = \frac{ECr}{rCL} = \frac{E}{L}$$

and

$$B_1 = \frac{ECL}{rCL} = \frac{E}{r}$$

Let

$$2\alpha = \frac{rC}{CL} = \frac{r}{L} \text{ and } \omega_0^2 = \frac{1}{CL}$$

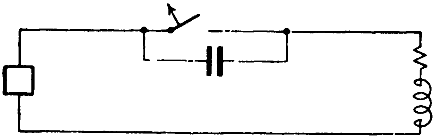

FIG. 24.

In both cases, $Z_{(p)} = 0$ gives $p = -\alpha \pm j\omega$ when oscillation conditions prevail $\left(\text{that is, when } \frac{1}{CL} > \frac{r^2}{4L^2}\right)$ and $\omega^2 = \omega_0^2 - \alpha^2$

$$p\frac{dZ}{dp} = 2p(p + \alpha)$$

and

$$\frac{Y_{(p)}}{p\frac{dZ}{dp}} = \frac{A_1}{2(p + \alpha)} \text{ for the first term}$$

and

$$\frac{pB_1}{2(p + \alpha)} \text{ for the second.}$$

Thus, $\left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p = p_1 = -\alpha + j\omega} = \frac{A_1}{2j\omega} = -j\frac{A_1}{2\omega}$ for the first term and $= \frac{B_1}{2\omega}(\omega + j\alpha)$ for the second.

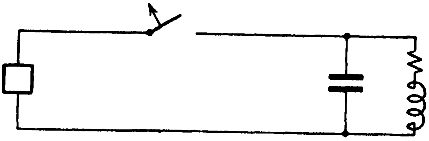

FIG. 25.

Referring then to equation (22), Chap. VIII,

A for the first term is $= \sqrt{0^2 + \frac{A_1^2}{4\omega^2}} = \frac{A_1}{2\omega} = \frac{E}{2L\omega}$

and A for the second term is $\frac{B_1}{2\omega}\sqrt{\omega^2 + \alpha^2} = \frac{E\omega_0}{2r\omega}$

For the first term, $\cot \delta = \dfrac{0}{-\dfrac{A_1}{2\omega}}$ or $\delta = -90°$

for the second term, $\tan \delta = \alpha/\omega$

$Y'_{(0)}/Z_{(0)}$ in both cases is zero; therefore,

$$i_c = \frac{E}{L\omega}\epsilon^{-\alpha t} \cos(\omega t - 90°) + \frac{E\omega_0}{r\omega}\epsilon^{-\alpha t} \cos(\omega t + \phi)$$

where $\tan \phi = \dfrac{\alpha}{\omega}$

$$= \frac{E}{L\omega} e^{-\alpha t} \cos(\omega t - 90°) + \frac{E\epsilon^{-\alpha t}\omega_0}{r\omega} \sin(\omega t + \phi_1)$$

where $\tan \phi_1 = \dfrac{\omega}{\alpha}$. For $t = 0$,

$$i_c = \frac{E\omega_0\omega}{r\omega\omega_0} = \frac{E}{r}$$

The condenser current is the entire original current at $t = 0$.

In solving this problem, the two following useful operators have been obtained:

$$\frac{A_1 p}{p^2 + 2\alpha p + \omega_0^2}\mathbf{1} \text{ and } \frac{B_1 p^2}{p^2 + 2\alpha p + \omega_0^2}\mathbf{1}$$

In Chap. XXIX is given a long list of useful operators.

CHAPTER XIII

HEAVISIDE'S "SHIFTING"

Heaviside writes (on p. 294, Vol. II)

$$f(p)u\epsilon^{at} = \epsilon^{at}f(p + a)u^{*} \tag{28}$$

where $f(p)$ is a function of p, usually written $Y_{(p)}/Z_{(p)}$, and u is some function of t. It may, for instance, be $\sin(\omega t + \alpha)$ or merely the unit function 1, depending upon the problem.

Proof.—First let $f(p)$ be p. Then the left-hand side of equation (28) becomes

$$p(u\epsilon^{at}) = \frac{d}{dt}(u\epsilon^{at}) = \epsilon^{at}\frac{du}{dt} + au\epsilon^{at}$$

$$= \epsilon^{at}\left[\frac{du}{dt} + au\right] = \epsilon^{at}(p + a)u$$

which satisfies the right-hand member of equation (28).

Similarly,

$$p^{n}(u\epsilon^{at}) = \epsilon^{at}(p + a)^{n}u$$

It is seen that equation (28) holds for $f(p) = p^{n}$. It will, hold therefore, for any function of p which can be expanded in a power series. It can be easily shown that the rule also applies for $1/p$.

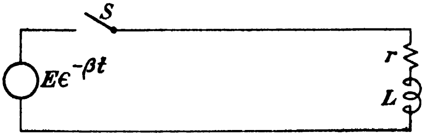

FIG. 26

To illustrate the use of this relation, assume that when switch S is closed in Fig. 26, a voltage $E\epsilon^{-\beta t}$ is applied to the inductive circuit. Find the voltage consumed by the inductance.

---

[*] It would, perhaps, have been clearer if it had been written $f(p)(u\epsilon^{\alpha t}) = \epsilon^{\alpha t}f(p + a)u$, because $f(p)$ operates on $u\epsilon^{\alpha t}$.

If unit e.m.f. had been impressed,

$$i = \frac{E\mathbf{1}}{r + pL}$$

and

$$e_t = pLi = \frac{pLE}{r + pL}\mathbf{1} = E\frac{p}{p + \alpha}\mathbf{1}, \text{ where } \alpha = \frac{r}{L}$$

Put

$$f(p) = \frac{p}{p + \alpha}$$

and

$$u\epsilon^{at} = E\epsilon^{-\beta t}\mathbf{1}$$

then

$$\frac{p}{p + \alpha} \text{ should operate on } E\epsilon^{-\beta t}\mathbf{1}$$

or

$$e_t = \frac{p}{p + \alpha}(E\epsilon^{-\beta t})\mathbf{1} \tag{29}$$

But we do not always know how to operate on a function of t, and, therefore, some transformation frequently must be made. The theorem given above says that $\epsilon^{-\beta t}$ can be shifted outside the operator if, in $f(p)$, that is in $\frac{p}{p + \alpha}$, p is changed to $p - \beta$.

$$\therefore e_t = E\epsilon^{-\beta t} \cdot \frac{p - \beta}{p - \beta + \alpha}\mathbf{1} = E\epsilon^{-\beta t}\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1}$$

By the use of the expansion theorem, we find that

$$\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1} = \frac{1}{\alpha - \beta}[\alpha\epsilon^{-(\alpha-\beta)t} - \beta]$$

$$\therefore e_t = \frac{E}{\alpha - \beta}[\alpha\epsilon^{-\alpha t} - \beta\epsilon^{-\beta t}] \tag{30}$$

As a matter of fact, in this simple case, this transformation was not necessary, because it is known that $(E\epsilon^{-\beta t})\mathbf{1}$ in equation (29) can be written $E\frac{p}{p + \beta}\mathbf{1}$. Therefore, the solution with every term in p is

$$e_t = \frac{p^2E}{(p + \alpha)(p + \beta)}\mathbf{1}$$

which could have been solved by the expansion theorem and would have given the solution shown in equation (30).

The shifting operation, of course, works both ways, and sometimes it is an advantage to reverse the operation. Thus, assume that we have an equation

$$\frac{p}{p-\alpha}\mathit{1} \tag{31}$$

This could be written

$$\epsilon^{-\beta t}\epsilon^{+\beta t}\frac{p}{p-\alpha}\mathit{1} \tag{32}$$

$\epsilon^{+\beta t}$ can be shifted to the right, that is, back of a new operator, if, in the new operator, p is changed to $p-\beta$.

Then equation (32) becomes

$$\epsilon^{-\beta t}\cdot\frac{p-\beta}{p-\beta-\alpha}(\epsilon^{+\beta t}\mathit{1}) \tag{33}$$

Since $\epsilon^{\beta t}\mathit{1} = \frac{p}{p-\beta}\mathit{1}$, equation (31) becomes

$$\epsilon^{-\beta t}\frac{p-\beta}{p-\beta-\alpha}\frac{p}{p-\beta}\mathit{1} = \epsilon^{-\beta t}\frac{p}{p-\beta-\alpha}\mathit{1}$$

Thus,

$$\frac{p}{p-\alpha}\mathit{1} = \epsilon^{-\beta t}\frac{p}{p-\alpha-\beta}\mathit{1}, \tag{34}$$

and for $\beta = \alpha$

$$\frac{p}{p-\alpha}\mathit{1} = \epsilon^{-\alpha t}\frac{p}{p-2\alpha}\mathit{1} \tag{35}$$

Note that in shifting to the left we retain the sign of the exponential, while in shifting to the right we reverse it. Thus

$$\frac{p}{p+\alpha}(\epsilon^{-\alpha t}\mathit{1}) = \epsilon^{-\alpha t}\frac{p-\alpha}{p+\alpha-\alpha}\mathit{1} = \epsilon^{-\alpha t}\frac{p-\alpha}{p}\mathit{1}$$

$$= \epsilon^{-\alpha t}\left(1-\frac{\alpha}{p}\right)\mathit{1} = \epsilon^{-\alpha t}(1-\alpha t)\mathit{1} \tag{36}$$

and

$$\epsilon^{-\alpha t}\frac{p}{p+\alpha}\mathit{1} = \frac{p+\alpha}{p+2\alpha}(\epsilon^{-\alpha t}\mathit{1}) = \frac{p+\alpha}{p+2\alpha}\cdot\frac{p}{p+\alpha}\mathit{1}$$

$$= \frac{p}{p+2\alpha}\mathit{1} = \epsilon^{-2\alpha t}\mathit{1} \tag{37}$$

$$\frac{1}{Z_{(p)}}\mathit{1} = \epsilon^{-\alpha t}\epsilon^{+\alpha t}\frac{1}{Z_{(p)}}\mathit{1} = \epsilon^{-\alpha t}\frac{1}{Z_{(p-\alpha)}}(\epsilon^{+\alpha t}\mathit{1})$$

$$= \epsilon^{-\alpha t}\frac{1}{Z_{(p-\alpha)}}\cdot\frac{p}{p-\alpha}\mathit{1} \tag{38}$$

For

$$\frac{1}{Z_{(p)}} = \frac{p}{\sqrt{p+\alpha}}, \text{ we get from equation (38)}$$

$$\frac{p}{\sqrt{p+\alpha}}\mathit{1} = \epsilon^{-\alpha t}\frac{p-\alpha}{\sqrt{p-\alpha+\alpha}}\,\frac{p}{p-\alpha}\mathit{1}$$

$$= \epsilon^{-\alpha t}p^{\frac{1}{2}}\mathit{1} = \frac{\epsilon^{-\alpha t}}{\sqrt{\pi t}}\mathit{1} \qquad (39)$$

This relation is explained in Chap. XXIII. Thus,

$$\frac{1}{\sqrt{p+\alpha}}\mathit{1} = \int_0^t \frac{\epsilon^{-\alpha t}\,dt}{\sqrt{\pi t}}$$

$$= 2\sqrt{\frac{t}{\pi}}\,\epsilon^{-\alpha t}\left(1 + \frac{2\alpha t}{1\cdot 3} + \frac{(2\alpha t)^2}{1\cdot 3\cdot 5} + \cdot\right) \qquad (40)$$

Another Example.—A certain problem gives the following operational solution:

$$y = \sqrt{\frac{p}{p+2\alpha}}\,\mathit{1} \qquad (41)$$

Find the numerical value of y.

The general method outlined in Chap. I would be to write

$$y = \sqrt{\frac{1}{1+\frac{2\alpha}{p}}}\,\mathit{1} = \left(\frac{1}{1+\frac{2\alpha}{p}}\right)^{\frac{1}{2}}\mathit{1}$$

and then to expand

$$\left(1+\frac{2\alpha}{p}\right)^{-\frac{1}{2}}\mathit{1}$$

in an infinite series; thus:

$$y = \left(1 - \frac{1}{2}\left(\frac{2\alpha}{p}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{2\alpha}{p}\right)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{2\alpha}{p}\right)^3 \cdots\right)\mathit{1}$$

$$= \left(1 - \alpha t + \frac{1\cdot 3\cdot 4}{2\cdot 4}\,\frac{\alpha^2 t^2}{2} - \cdots\right)\mathit{1}$$

$$= \left(1 - \alpha t + \frac{1\cdot 3(\alpha t)^2}{(\underline{|2})^2} - \frac{1\cdot 3\cdot 5(\alpha t)^3}{(\underline{|3})^2} \cdots\right)\mathit{1} \qquad (42)$$

Equation (41) may be written

$$y = \left(\frac{p}{p+2\alpha}\right)^{\frac{1}{2}}\mathit{1} = \epsilon^{-\alpha t}\epsilon^{+\alpha t}\left(\frac{p}{p+2\alpha}\right)^{\frac{1}{2}}\mathit{1}$$

Now shift $\epsilon^{\alpha t}$ to the right and change operator. Substituting for p, $p - \alpha$,

$$
\begin{aligned}
\therefore y &= \epsilon^{-\alpha t}\left(\frac{p-\alpha}{p+\alpha}\right)^{\frac{1}{2}}(\epsilon^{+\alpha t}\,\mathit{1}) \\
&= \epsilon^{-\alpha t}\left(\frac{p-\alpha}{p+\alpha}\right)^{\frac{1}{2}}\frac{p}{p-\alpha}\,\mathit{1} = \epsilon^{-\alpha t}\frac{p}{\sqrt{(p+\alpha)(p-\alpha)}}\,\mathit{1} \\
&= \epsilon^{-\alpha t}\frac{p}{(p^2-\alpha^2)^{\frac{1}{2}}}\,\mathit{1} = \epsilon^{-\alpha t}\frac{1}{\left(1-\frac{\alpha^2}{p^2}\right)^{\frac{1}{2}}}\,\mathit{1} \qquad (43)
\end{aligned}
$$

This gives

$$
\begin{aligned}
y &= \epsilon^{-\alpha t}\left|1 + \frac{1}{2}\frac{\alpha^2}{p^2} + \frac{1\cdot 3}{2\cdot 4}\frac{\alpha^4}{p^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{\alpha^6}{p^6} + \cdots\right|\mathit{1} \\
&= \epsilon^{-\alpha t}\left|1 + \frac{1}{2}\frac{\alpha^2 t^2}{2!} + \frac{1\cdot 3\alpha^4 t^4}{2\cdot 4\;4!} + \frac{1\cdot 3\cdot 5\alpha^6 t^6}{2\cdot 4\cdot 6\;6!} + \right|\mathit{1} \\
&= \epsilon^{-\alpha t}\left[1 + \frac{(\alpha t/2)^2}{1!^2} + \frac{(\alpha t/2)^4}{2!^2} + \frac{(\alpha t/2)^6}{3!^2} + \cdots\right|\mathit{1} \\
&= \epsilon^{-\alpha t}J_0(i\alpha t)\,\mathit{1} \qquad (44)
\end{aligned}
$$

$J_0\,(i\alpha t)$ is the Bessel function of the 0th order with the imaginary argument $i\alpha t$. Heaviside uses the notation $I_0(\alpha t)$.

In order to illustrate a point that may cause some confusion, another example will be given.

Consider an inductive circuit to which is applied a voltage $E\epsilon^{-\beta t}\cos\omega t$ at $t = 0$. The conventional relation would then be:

$$
E\epsilon^{-\beta t}\cos\omega t = ir + L\frac{di}{dt} = i(r + pL)
$$

It has been shown in Chap. VI that $\cos\omega t = \frac{p^2}{p^2+\omega^2}\mathit{1}$. Thus,

$$
i = \frac{E}{r+pL}\left(\epsilon^{-\beta t}\frac{p^2}{p^2+\omega^2}\mathit{1}\right)
$$

Here we desire to shift $\epsilon^{-\beta t}$ outside, and the rule was to do so while at the same time p is changed to $p - \beta$. Thus, we write

$$
i = E\epsilon^{-\beta t}\frac{1}{r + L(p-\beta)}\cdot\frac{p^2}{p^2+\omega^2}\mathit{1}
$$

The procedure is then to solve for

$$\frac{1}{r + L(p - \beta)} \cdot \frac{p^2}{p^2 + \omega^2}\mathbf{1}$$

and then multiply the result by $E\epsilon^{-\beta t}$ to get i. $Z_{(p)} = 0$ gives three roots: $p_1 = \beta - \frac{r}{L}$, $p_2 = +j\omega$, and $p_3 = -j\omega$, so that $i_1 = A\epsilon^{\left(\beta - \frac{r}{L}\right)t} + B\epsilon^{j\omega t} + C\epsilon^{-j\omega t}$. The last two terms combine to the permanent term as described in Chap. VI. The coefficient for the remaining term is $\frac{Y_{(p)}}{ph_2(p)h_1'(p)}$ for $p = p_1$ where $h_2(p) = p^2 + \omega^2$, and $h_1(p) = r - L\beta + pL$, and $Y_{(p)} = p^2$

$$\therefore A = \frac{p^2}{pL(p^2 + \omega^2)} = \frac{p}{L(p^2 + \omega^2)}.$$

For $p = p_1$ it is

$$\frac{\beta - \frac{r}{L}}{L\left[\left(\beta - \frac{r}{L}\right)^2 + \omega^2\right]} = \frac{\beta L - r}{(\beta L - r)^2 + \omega^2 L^2}$$

The permanent term of i_1 corresponds to an impedance $(r - \beta L + j\omega L)$ or

$$Z = \sqrt{(r - \beta L)^2 + \omega^2 L^2}$$

$$\therefore i_1 = \frac{E}{Z}\cos(\omega t - \delta) + \frac{E(\beta L - r)\epsilon^{\left(\beta - \frac{r}{L}\right)t}}{(\beta L - r)^2 + \omega^2 L^2}$$

where

$$\tan \delta = \frac{\omega L}{r - \beta L}$$

Therefore,

$$i = \frac{E\epsilon^{-\beta t}}{Z}\cos(\omega t - \delta) - E\frac{r - \beta L}{Z^2}\epsilon^{-\frac{r}{L}t}$$

$$= \frac{E}{Z}\left[\epsilon^{-\beta t}\cos(\omega t - \delta) - \epsilon^{-\frac{r}{L}t}\cos\delta\right]$$

The same result could, of course, have been obtained without "shifting" by the use of the operator for $\epsilon^{-\beta t}\cos \omega t$ given in Chap. VI. In that case, the operational solution is

$$i = \frac{p(p + \beta)}{(p + \beta)^2 + \omega^2} \cdot \frac{1}{r + pL}\mathbf{1}.$$

CHAPTER XIV

DUHAMEL'S INTEGRAL

The use of Duhamel's integral is often very advantageous when dealing with an impressed e.m.f. which is not of "unit-function" shape. It is especially convenient when dealing with problems involving "distributed" constants, as is the case in transmission-line or heat calculations.

The procedure then is: Find, first, the solution in the case of unit function; let this be $\varphi(t)$. Let $\varphi(0)$ be the value of $\varphi(t)$ when $t = 0$. Let the impressed e.m.f. be $e(t)$, and let $e(u)$ be $e(t)$ when u is substituted for t Then one form of the solution is

$$e(t)\varphi(0) + \int_{u=0}^{u=t} e(u) \frac{d}{d(t-u)} \varphi(t-u)du \qquad (45)$$

Another form is given in equation (46).

Before giving the proof of equation (45), it may be well to illustrate its use in a particular case. Find the current if an e.m.f. $e(t) = E \sin(\omega t + \varphi)$ is impressed on an inductive circuit as shown in Fig. 27.

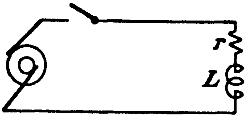

Fig. 27.

The solution with unit e.m.f. is

$$i = \frac{1}{r + pL}1 = \frac{1}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

thus,

$$\varphi(t) = \frac{1}{r}\left[1 - \epsilon^{-\frac{r}{L}t}\right]$$

$$\varphi(0) = 0$$

$$\varphi(t-u) = \frac{1}{r}\left[1 - \epsilon^{-\frac{r}{L}(t-u)}\right]$$

$$\frac{d}{d(t-u)}\varphi(t-u) = \frac{1}{L}\epsilon^{-\frac{r}{L}(t-u)}$$

$$e(u) = E \sin(\omega u + \varphi)$$

$$\therefore i = \int_{u=0}^{u=t} E \frac{1}{L} \epsilon^{-\frac{r}{L}(t-u)} du \sin(\omega u + \varphi)$$

$$= \frac{E}{L} \epsilon^{-\frac{r}{L}t} \int_{u=0}^{u=t} \epsilon^{\frac{r}{L}u} \sin(\omega u + \varphi) du$$

$$= \frac{E}{L}\left[\frac{\sin(\omega t + \varphi - \delta) - \epsilon^{-\frac{r}{L}t} \sin(\varphi - \delta)}{\sqrt{\frac{r^2}{L^2} + \omega^2}}\right]$$

$$= \frac{E}{Z}\left[\sin(\omega t + \varphi - \delta) - \epsilon^{-\frac{r}{L}t} \sin(\varphi - \delta)\right]$$

where

$$Z = \sqrt{r^2 + \omega^2 L^2}, \text{ and } \tan\delta = \frac{\omega L}{r}$$

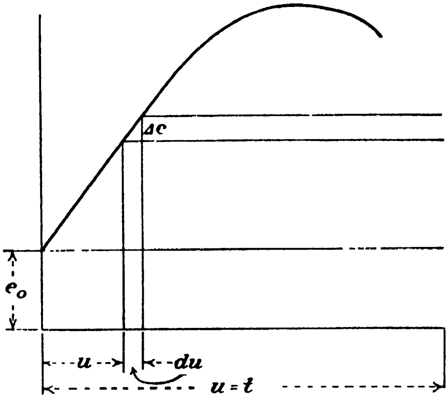

FIG. 27a.

The explanation of this integral, which was given almost one hundred years ago by Duhamel, is perhaps best made by referring to Fig. 27a. Let the e.m.f. be represented by the curved line. The curved line can be approached by assuming that at $t = 0$ an e.m.f. e_0 suddenly is impressed. A short time afterward, another rectangular e.m.f. e_1 is added; sometime after, an e.m.f. e_2, etc.; all these being of "unit" shape. The current at time t is then the sum of the currents due to e_0 beginning at $t = 0$, e_1 beginning at $t = u_1$, etc. The current due to e_0 is, of course, $e_0\,\varphi(t)$ where $\varphi(t)$ is the solution with unit e.m.f. impressed at $t = 0$.

The current due to the small e.m.f. Δe, which begins at time u, is obviously $\Delta e \cdot \phi(t - u)$, $t - u$ being the time elasped since the switching of unit Δe. Therefore, the total current at $t = t$, is

$$i = e_0\varphi(t) + \sum_{u=0}^{u=t} \Delta e\, \phi(t - u)$$

But

$$\Delta e = e(u + du) - e(u)$$

and thus

$$\frac{\Delta e}{du} = \frac{e(u + du) - e(u)}{du} = \frac{d}{du}e(u)$$

Therefore,

$$\Delta e = \frac{d}{du}e(u)du$$

$$i = e_0\varphi(t) + \int_{u=0}^{u=t} \varphi(t - u)\frac{d}{du}e(u)du \qquad (46)$$

This, then, is one solution which is sometimes convenient to use. The integral may be written

$$\int_0^t UdV = \Big[UV\Big]_0^t - \int^t VdU$$

Let

$$U = \varphi(t - u)$$

and

$$dV = de(u)$$

or

$$V = e(u)$$

Then

$$\int_0^t \varphi(t - u)\frac{d}{du}e(u)du$$

$$= \Big[e(u)\varphi(t - u)\Big]_{u=0}^{u=t} - \int_{u=0}^{u=t} e(u)\frac{d}{du}\varphi(t - u)du$$

$$= e(t)\varphi(0) - e(0)\varphi(t) + \int_{u=0}^{u=t} e(u)\frac{d}{d(t - u)}\varphi(t - u)du$$

and the new expression of equation (46) becomes

$$i = e(0)\varphi(t) + e(t)\varphi(0) - e(0)\varphi(t) + \int_{u=0}^{u=t} e(u)\frac{d}{d(t - u)}\phi(t - u)du$$

$$= e(t)\varphi(0) + \int_{u=0}^{u=t} e(u)\frac{d}{d(t - u)}\varphi(t - u)du,$$

which was to be proved.

There are, thus, in general, three ways in which problems can be solved when an arbitrary e.m.f. is impressed upon a circuit:

1. It may be possible to convert the e.m.f. into an operator on $\mathbf{1}$.
2. If the arbitrary e.m.f. contains an exponential, Heaviside's "shifting" may be used.
3. Duhamel's integral may be used.

To illustrate the three methods, consider an e.m f. $E\epsilon^{-\beta t}\mathbf{1}$ impressed on a circuit of resistance and capacity in series: find the current.

The resistance operator is $r + \frac{1}{pC} = \frac{pCr+1}{pC}$

Method 1.—Since $E\epsilon^{-\beta t}\mathbf{1} = E\frac{p}{p+\beta}\mathbf{1}$, the operational solution is then

$$i = \frac{EpC}{pCr+1} \times \frac{p}{p+\beta}\mathbf{1} = EC\frac{p^2}{(pCr+1)(p+\beta)}\mathbf{1}$$

which, solved by the expansion theorem, gives

$$i = \frac{EC}{1-\beta Cr}\left(\frac{\epsilon^{-\frac{t}{Cr}}}{Cr} - \beta\epsilon^{-\beta t}\right)$$

Method 2:

$$i = \frac{pC}{pCr+1}(\epsilon^{-\beta t}E)\mathbf{1}$$

Shift $\epsilon^{-\beta t}$ outside the operator and get

$$i = EC\epsilon^{-\beta t}\frac{p-\beta}{(p-\beta)Cr+1}\mathbf{1} = EC\epsilon^{-\beta t}\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1}$$

$\frac{Y_{(p)}}{Z_{(p)}}$ by the expansion theorem is

$$\frac{1}{1-Cr\beta}\left[\frac{\epsilon^{\left(\beta-\frac{1}{Cr}\right)t}}{Cr} - \beta\right]$$

$$\therefore i = \frac{EC}{1-Cr\beta}\left[\frac{\epsilon^{-\frac{1}{Cr}t}}{Cr} - \beta\epsilon^{-\beta t}\right]$$

Method 3.—The solution with unit operator is $i = \frac{pCE}{pCr+1}\mathbf{1}$

$$\therefore i = \frac{E}{r}\epsilon^{-\frac{1}{Cr}t}$$

Put

$$\varphi(t) = \frac{1}{r}\epsilon^{-\frac{1}{Cr}t}.$$

Then

$$\varphi(t-u) = \frac{1}{r}\epsilon^{-\frac{1}{Cr}(t-u)}$$

$$\frac{d}{d(t-u)}\varphi(t-u) = -\frac{1}{Cr^2}\epsilon^{-\frac{1}{Cr}(t-u)}$$

$$e(u) = E\epsilon^{-\beta u}, \quad e(t) = E\epsilon^{-\beta t} \quad \text{and } \varphi(0) = \frac{1}{r}$$

$$\therefore i = E\frac{\epsilon^{\beta t}}{r} + \int_{u=0}^{u=t} -E\frac{\epsilon^{-\beta u}}{Cr^2}\epsilon^{-\frac{1}{Cr}(t-u)}du$$

$$= E\frac{\epsilon^{-\beta t}}{r} - \frac{E}{Cr^2}\epsilon^{-\frac{1}{Cr}t}\int_{u=0}^{u=t}\epsilon^{\left(\frac{1}{Cr}-\beta\right)u}du$$

$$= \frac{EC}{1-Cr\beta}\left[\frac{\epsilon^{-\frac{1}{Cr}t}}{Cr} - \beta\epsilon^{-\beta t}\right]$$

In connection with the use of Duhamel's integral, certain expressions seem to appear frequently. It is, therefore, advantageous to record them for reference.

$$\epsilon^{-\alpha t}\int_{u=0}^{u=t}\epsilon^{\alpha u}\sin(\omega u+\delta)du$$

$$= \frac{\sin(\omega t+\delta-\varphi) - \epsilon^{-\alpha t}\sin(\delta-\varphi)}{\sqrt{\alpha^2+\omega^2}}$$

$$\epsilon^{-\alpha t}\int_{u=0}^{u=t}\epsilon^{\alpha u}\cos(\omega u+\delta)du$$

$$= \frac{\cos(\omega t+\delta-\varphi) - \epsilon^{-\alpha t}\cos(\delta-\varphi)}{\sqrt{\alpha^2+\omega^2}}$$

where

$$\tan\varphi = \frac{\omega}{\alpha}$$

or

$$\epsilon^{-\alpha t}\int_{u=0}^{u=t}\epsilon^{\alpha u}\sin(\omega u+\delta)du$$

$$= \frac{-\cos(\omega t+\delta+\psi) + \epsilon^{-\alpha t}\cos(\delta+\psi)}{\sqrt{\alpha^2+\omega^2}}$$

and

$$\epsilon^{-\alpha t}\int_{u=0}^{u=t}\epsilon^{\alpha u}\cos(\omega u+\delta)du$$
$$=\frac{\sin(\omega t+\delta+\psi)-\epsilon^{-\alpha t}\sin(\delta+\psi)}{\sqrt{\alpha^2+\omega^2}}$$

where

$$\tan\psi = \alpha/\omega$$

CHAPTER XV

GENERAL EQUATIONS PERTAINING TO TRANSMISSION LINES. OPEN AND SHORT-CIRCUIT CONDITIONS OF "IDEAL" CABLE

Heaviside devotes only a few pages out of about one thousand to circuits of concentrated resistance, inductance, capacity, and mutual inductance. The remaining pages deal largely with cable problems and more particularly with "ideal cables" where the numerical work becomes fairly simple.

Consider a transmission line as shown in Fig. 28.

Let the constants per unit length of transmission distance be R, L, C, G.

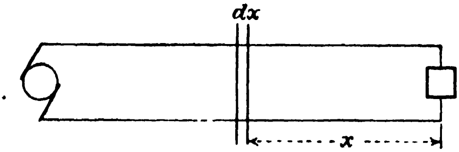

FIG. 28.

The well-known relations between current and e.m.f. at any point, distant x from the end, is

$$\frac{\partial i}{\partial x}dx = eGdx + Cdx\frac{\partial e}{\partial t} \tag{47}$$

and

$$\frac{\partial e}{\partial x}dx = iRdx + Ldx\frac{\partial i}{\partial t}.^{*} \tag{48}$$

From these we derive

$$\frac{\partial i}{\partial x} = Ge + C\frac{\partial e}{\partial t} = Ge + pCe = e(G + pC) \tag{49}$$

$$\frac{\partial e}{\partial x} = Ri + pLi = i(R + pL) \tag{50}$$

---

[*] These equations simply state that the difference in current on each side of the line element dx is the leakage current and the charging current of the element. Similarly, the difference in voltage is that consumed by the resistance and the inductance of the element

and from these,

$$\frac{\partial^2 e}{\partial x^2} = (R + pL)(G + pC)e = n^2 e \tag{51}$$

where n is a function of p but not of x. The solution of this equation is well known. It is

$$e = A_1 \epsilon^{nx} + A_2 \epsilon^{-nx} \tag{52}$$
$$= A_1(\cosh nx + \sinh nx) + A_2(\cosh nx - \sinh nx)$$
$$= (A_1 + A_2) \cosh nx + (A_1 - A_2) \sinh nx$$
$$e = K_1 \cosh nx + K_2 \sinh nx \tag{53}$$

where K_1 and K_2 are determined from the terminal conditions.

Since from equation (49)

$$\frac{di}{dx} = (G + pC)e = Ye$$

it follows that

$$i = Y \int e dx$$

$$\therefore i = \frac{Y}{n}[K_1 \sinh nx + K_2 \cosh nx] \tag{54}$$

For these equations,

$$Y = G + pC$$

and in subsequent equations we shall write $Z = R + pL$ so that $n^2 = YZ$; therefore $Y/n = n/Z$ (55)

In the equations given below the Y/n form has been used. The other form n/Z is often preferable in determining the roots corresponding to $Z(p) = 0$.

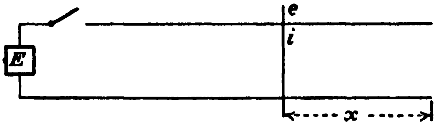

FIG. 29.

Special Cases.—1. *Open Line (Fig. 29):* Then

$$i = 0 \text{ for } x = 0;\ e = E \text{ for } x = l$$
$$\therefore E = K_1 \cosh nl + K_2 \sinh nl$$
$$0 = \frac{Y}{n}[K_1 \sinh 0 + K_2 \cosh 0]$$

Since cosh 0 is 1, it is evident that K_2 must be zero.

$$\therefore e = K_1 \cosh nx$$

$$\therefore E = K_1 \cosh nl, \text{ or } K_1 = \frac{E}{\cosh nl}$$

and

$$\left.\begin{aligned} e &= E\frac{\cosh nx}{\cosh nl}1 \\ &\text{and} \\ i &= \frac{Y}{n} \times E\frac{\sinh nx}{\cosh nl}1 \end{aligned}\right\} \begin{array}{l}\text{under} \\ \text{Heaviside} \\ \text{conditions}\end{array} \qquad (56)$$

The method of solving equations of this type by the expansion theorem will be shown later.

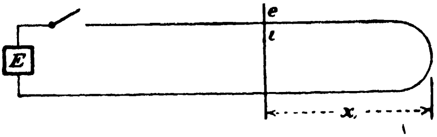

Fig. 30.

2. *Short-circuited Line* (*Fig.* 30):

For $x = 0$, the potential difference $= 0$. Therefore, $e = 0$ for $x = 0$. For $x = l$, $e = E$

$$\therefore 0 = K_1 \cosh 0 + K_2 \sinh 0$$

$$\therefore K_1 = 0$$

and

$$e = K_2 \sinh nx. \quad \text{For } x = l,\ e = E$$

$$\therefore K_2 = \frac{E}{\sinh nl}$$

$$\left.\begin{aligned} \therefore e &= E\frac{\sinh nx}{\sinh nl}1 \\ i &= \frac{Y}{n}E\frac{\cosh nx}{\sinh nl}1 \end{aligned}\right\} \begin{array}{l}\text{under} \\ \text{Heaviside} \\ \text{conditions}\end{array} \qquad (57)$$

It is not difficult to solve equations (56) and (57) in the most general case, that is, with transmission lines having resistance, inductance, capacity, and leakage, as will be shown in Chap. XVII.

At first in order to show more clearly the method employed, the application will be made to what Heaviside calls an *ideal cable*. This is a cable which has insignificant leakage and inductance and, therefore, only two constants—resistance and capacity per unit length.

No such cable can exist, but many cables closely approach this condition. The most serious objection to the study of such cable is that the equations do not show the velocity of propagation of the current and the voltage waves. The advantages are that the results obtained are satisfactory for all practical purposes and that the mathematical transformations are few.

Open-circuited Case.—Referring to the open-circuited case and letting Fig. 29 represent such an *ideal cable*—or transmission line—we have, from equation (56),

$$e = \frac{E \cosh nx}{\cosh nl} 1$$

and

$$i = E\frac{Y}{n}\frac{\sinh nx}{\cosh nl} 1$$

The problem is to solve these operational relations by the expansion theorem. While it is not necessary, it is usually best to introduce trigonometric relations rather than hyperbolic. Engineers, at any rate, are far more familiar with trigonometric functions than with hyperbolic functions.

In Chap. XXIX, it will be shown that $\cosh ja = \cos a$, and $\sinh ja = j \sin a$. If, therefore, we write $n^2 = -m^2$ then $n = \pm jm$ and

$$\cosh nx = \cosh (\pm jmx) = \cos mx$$

and

$$\sinh nx = \sinh (\pm jmx) = \pm j \sin mx$$

$$\therefore e = E\frac{\cos mx}{\cos ml} 1$$

$Z_{(p)} = 0$ gives $\cos ml = 0$; thus, $ml = (\pm \pi/2, \pm 3\pi/2, \pm 5\pi/2)$ or $\pm\frac{2s-1}{2}\pi$ and

$$m = \pm\frac{2s-1}{2}\frac{\pi}{l} \tag{58}$$

where the first value of s is 1, the next 2, etc. We could also have written $ml = \pm\frac{2s+1}{2}\pi$, in which case the first term corresponds to $s = 0$, the next to $s = 1$, etc.

The question may be raised: Is it permissible to use the expansion theorem? Perhaps one of the roots of p is zero. If so, $Y_{(p)}/Z_{(p)}$ would be infinity when we substitute $p = 0$. To test it, we have

$$n^2 = (R + pL)(G + pC)$$

for $p = 0$

$$n^2 = RG \text{ or } n = \pm\sqrt{RG}$$

$$\therefore \frac{Y_{(p)}}{Z_{(p)}} = \frac{\cosh x\sqrt{RG}}{\cosh l\sqrt{RG}} \text{ for } p = 0$$

This is not infinity.

In our case, $G = 0$, and thus $Y_{(p)}/Z_{(p)} = \cosh 0/\cosh 0 = 1$, for $p = 0$. Thus, we conclude that the expansion theorem can be used. Let the problem be to find the voltage difference between the two lines and the current flowing at any point x (measured from the end of the line) at any time t after a storage battery of voltage E has been connected to this "ideal" open-circuited cable.

$$\frac{dZ}{dp} = \frac{dZ}{dm}\frac{dm}{dp}$$

but, from equation (51), $-m^2 = pCR$, and thus $-2mdm = CRdp$, or $\frac{dm}{dp} = -\frac{CR}{2m}$. Also, $p = -\frac{m^2}{CR}$ and $\frac{dZ}{dm} = -l\sin ml$. Therefore,

$$p\frac{dZ}{dp} = \left(-\frac{m^2}{CR}\right)(-l\sin ml)\left(-\frac{CR}{2m}\right) = -\frac{ml}{2}\sin ml \quad (59)$$

Note that since $p = -\frac{m^2}{CR}$, there is only one value of p corresponding to the two values of m, that is, corresponding to the roots $\pm\frac{2s-1}{2}\frac{\pi}{l}$. It must, of course, be remembered that the Heaviside expansion theorem is a summation over the roots of p and not the roots of m.

After this digression, we return to equation (59) and note that for $s = 1$, $\sin ml = \sin\frac{\pi}{2} = 1$; for $s = 2$, $\sin ml = \sin\frac{3\pi}{2} = -1$; etc. Thus,

$$-\frac{ml}{2}\sin ml = \frac{ml}{2}(-1)^s \quad (60)$$

Thus,

$$e = E\left[1 + \sum_{s=1}^{s=\infty} \frac{2(-1)^s}{ml} \cos mx \cdot \epsilon^{-\frac{m^2}{CR}t}\right]$$

Since $Y_{(0)}\ Z_{(0)}$ has been shown to be equal to 1, we get, when substituting the value for m,

$$e = E\left[1 + \frac{4}{\pi}\sum_{s=1}^{s=\infty} (-1)^s \frac{\cos \frac{2s-1}{2} \cdot \frac{\pi x}{l}}{2s-1} \epsilon^{-\frac{(2s-1)^2\pi^2}{4CRl^2}t}\right] \qquad (61)$$

To find the current, substitute the values of Y and n in equation (56).

$$\frac{Y}{n} = \frac{n}{Z} \text{ since } n^2 = YZ, \text{ thus } \frac{Y}{n} = \frac{n}{Z} = \frac{\pm jm}{R}$$

$$\therefore i = -E\frac{m \sin mx}{R \cos ml} 1$$

$p\frac{dZ}{dp}$, in this case, obviously becomes $\frac{Rml}{2}(-1)^s$, $\frac{Y_{(0)}}{Z_{(0)}} = 0$. Thus,

$$i = \sum_{s=1}^{s=\infty} -\frac{2E}{Rl}(-1)^s \sin mx\ \epsilon^{-\frac{m^2}{CR}t} \qquad (62)$$

where m is defined by equation (58), using the positive sign only At the battery, $x = l$. $\therefore e = E$ for all values of t, and

$$i = \frac{2E}{Rl}\left[\epsilon^{-\frac{\pi^2}{4CRl^2}t} + \epsilon^{-\frac{9\pi^2}{4CRl^2}t} + \epsilon^{-\frac{25\pi^2}{4CRl^2}t} \cdots\right] \qquad (63)$$

At $t = 0$ and $x = l$, equation (62) is non-uniformly convergent, and, hence, equation (63) cannot be used at $t = 0$.

Equation (63) converges very rapidly with the smallest values of t, so that we can write, with a very fair degree of accuracy,

$$i = \frac{2E}{Rl}\epsilon^{-\frac{\pi^2}{4CRl^2}t}$$

If l is large, the current, which is initially large, persists for a considerable time; if l is short, the current dies down rapidly.

Thus, in the case of a short "ideal" line, the battery current is

$$i = \frac{2E}{R_0}\epsilon^{-\frac{\pi^2}{4R_0C_0}t}$$

where R_0 and C_0 are the *total* line resistance and capacity.

In a circuit consisting of concentrated resistance and capacity, the current is

$$i = \frac{E}{R_0}\epsilon^{-\frac{1}{C_0R_0}t}$$

It is thus evident that an "ideal" line or cable may, with fair degree of accuracy, be represented by placing the line capacity in the middle of the line. The initial current is, for all practical purposes,

$$i = \frac{2E}{Rl}$$

Short-circuited Case.—Consider, next, a short-circuited "ideal" cable as is represented in Fig. 30.

Substituting m for n, as in the previous case, equations (57) become

$$\left.\begin{aligned} e &= E\frac{\sin mx}{\sin ml}\mathbf{1} \\ i &= E\frac{m}{R}\frac{\cos mx}{\sin ml}\mathbf{1} \end{aligned}\right| \qquad (64)$$

$$Z_{(p)} = 0 \text{ gives } ml = 0, \pi, 2\pi \cdots \text{ or } \frac{s\pi}{l} \text{ or } m = \pm\frac{s\pi}{l} \qquad (65)$$

where

$$s = 0 \qquad s = 1 \qquad s = 2, \text{ etc.}$$

Again, we have to question whether $p = 0$ is one root. $n^2 = (R + pL)\ (G + pC)$. As p approaches zero, n also approaches zero, in the "ideal" cable,

$$\therefore \frac{\sinh nx}{\sinh nl} = \frac{nx}{nl} = \frac{x}{l}$$

which is not infinity. Therefore, $p = 0$ is not a root and the expansion theorem can be used. From equation (65), $m = 0$ when $s = 0$, and with this value we get the constant term $Y_{(0)}/Z_{(0)} = x/l$. Therefore, when using $Y_{(0)}/Z_{(0)}$ we should begin the series with $s = 1$.

In solving for e,

$$\frac{dZ}{dp} = \frac{dZ}{dm}\frac{dm}{dp} = (l \cos ml)\left(-\frac{CR}{2m}\right)$$

and

$$p\frac{dZ}{dp} = \left(-\frac{m^2}{CR}\right)(l \cos ml)\left(-\frac{CR}{2m}\right) = \frac{ml \cos ml}{2}$$

$$\therefore e = E\left[\frac{x}{l} + \sum_{s=1}^{\infty}\frac{2 \sin mx\, \epsilon^{-\frac{m^2}{CR}t}}{ml \cos ml}\right]$$

or since $\cos ml = -1, +1, -1$, etc., and $m = s\pi/l$

$$e = E\left[\frac{x}{l} + 2\sum_{s=1}^{\infty}(-1)^s\frac{\epsilon^{-\frac{s^2\pi^2}{CRl^2}t}\sin\frac{s\pi x}{l}}{s\pi}\right] \qquad (66)$$

By a similar calculation, the current is found to be

$$i = \frac{E}{Rl}\left[1 + 2\sum_{s=1}^{\infty}(-1)^s\epsilon^{-\frac{s^2\pi^2}{CRl^2}t}\cos\frac{s\pi x}{l}\right] \qquad (67)$$

At the end of the line, $x = 0$, and there

$$i_1 = \frac{E}{Rl}\left[1 + 2\sum_{s=1}^{\infty}(-1)^s\epsilon^{-\frac{s^2\pi^2}{CRl^2}t}\right]$$

In Fig. 31 (curve A) are shown the values of the current at the end of such a short-circuited cable. It also represents the current quite accurately, if the receiving device is of small resistance compared with that of the cable.

The unit of time taken is $T = \frac{2l^2CR \log \frac{4}{3}}{\pi^2}$. This particular value was suggested by Kelvin and gives a rapidly converging series.

The exponential term becomes

$$\epsilon^{-\frac{s^2\pi^2}{CRl^2}\frac{2l^2CR}{\pi^2}\log\frac{4}{3}} = \epsilon^{-2s^2\log\frac{4}{3}} = \left(\tfrac{3}{4}\right)^{2s^2}$$

$$\therefore i_1 = \frac{E}{Rl}\left[1 + 2\sum_{s=1}(-1)^s\left(\frac{3}{4}\right)^{2s^2}\right]$$

The particular values used pertain to a telephone cable No. 19 B. & S., 100 miles long where

R (per loop mile) = 88 ohms
C (per loop mile) = 0.054 microfarad

The unit of time T becomes, then, 2.77/1,000 seconds.

Curve C shows the shape of the current if the battery had been connected to the cable during a time interval T after which the cable had been grounded at the battery. Curves D, E, F, etc., show the received current if the battery had been connected during intervals of $2T$, $3T$, etc., before grounding.

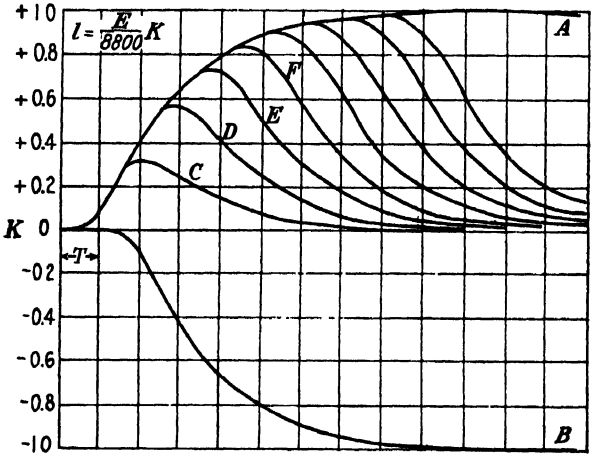

FIG. 31.—Current at receiving end of closed No. 19 B. & S. telephone cable

Curve A, current due to applying E at $t = 0$.

Curve B, current due to $-E$ after an interval T.

Curve C, actual current which is the sum of the two.

Curves D, E, F, etc., resultant current when the battery is grounded after intervals $2T$, $3T$, $4T$, etc.

Constants of cable.

C = 0.054 microfarad per mile of distance.

R = 88 ohms per mile distance.

T = unit of time taken as the "retardation."

$$T = \frac{2l^2CR}{\pi^2}\log\frac{4}{3}.$$

$$T = \frac{2.77}{1,000} \text{ for } l = 100 \text{ miles.}$$

The method of arriving at these curves should be apparent from the discussion in Chap. X.

Curve C is the sum of the original current beginning at $t = 0$ and a reversed similar current beginning at $t = T$, as shown in curve B.

CHAPTER XVI

AN ALTERNATOR SUDDENLY CONNECTED TO AN "IDEAL CABLE"

Let the problem be to find the value of the voltage at any time and place on a short-circuited cable (Fig. 32), if the impressed e.m.f. is $E \cos \omega t$ and if the switch is closed at $t = 0$.

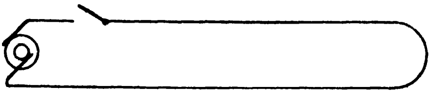

Fig. 32.

We may proceed by either of two routes—by the use of operator $\frac{p^2}{p^2 + \omega^2}\mathbf{1} = \cos \omega t$ (from Chap. VI) or by the use of Duhamel's integral (from Chap. XIII).

In the first case,

$$e = E\frac{\sinh nx}{\sinh nl}\,\frac{p^2}{p^2 + \omega^2}\mathbf{1} \text{ or } E\frac{\sin mx}{\sin ml}\,\frac{p^2}{p^2 + \omega^2}\mathbf{1} \qquad (68)$$

$Z_{(p)} = 0$ gives two roots $\pm j\omega$ and a series of roots $m = s\pi/l$, as discussed in Chap. XV. The permanent term corresponds to the two roots $\pm j\omega$, and it is usually simplest to determine that term in vector form by introducing $p = j\omega$ in the operational solution for the unit e.m.f., as has been shown in the footnote, Chap. IV.

$$e = E\frac{\sinh nx}{\sinh nl}, \text{ but } n^2 = pCR = j\omega CR$$

Thus,

$$n = \sqrt{j}\sqrt{\omega CR} = \sqrt{\omega CR}\,\underline{|45^\circ} = \sqrt{\frac{\omega CR}{2}}(1 + j) = a(1 + j)$$

Therefore,

$$e = E\frac{\sinh\,[ax(1 + j)]}{\sinh\,[al(1 + j)]}$$

but the magnitude of sinh $(a + jb)$ is $\sqrt{\sinh^2 a + \sin^2 b}$ and the slope is obtained from the relation $\tan \varphi = \coth a \tan b$ (see Chap. XXIX). Thus,

$$\underset{\cdot}{e} = E\sqrt{\frac{\sinh^2 ax + \sin^2 ax}{\sinh^2 al + \sin^2 al}}\,\big|\varphi_1 - \varphi_2$$

where $\tan \varphi_1 = \coth ax \tan ax$ and $\tan \varphi_2 = \coth al \tan al$ or

$$e_p = E\sqrt{\frac{\sinh^2 ax + \sin^2 ax}{\sinh^2 al + \sin^2 al}} \cos(\omega t + \varphi_1 - \varphi_2)$$

To calculate the transient term, it is best to resort to the method shown in Chap. VI. Let

$$h_{1(p)} = \sin ml \text{ and } h_{2(p)} = p^2 + \omega^2$$

It is then necessary to find

$$\frac{Y_{(p)}}{ph_{2(p)}\,h'_{1(p)}}$$

$$ph'_{1(p)} = pl \cos ml\frac{dm}{dp}, \text{ or since } -m^2 = pCR$$

$$ph'_{1(p)} = \frac{ml}{2} \cos ml$$

and

$$ph_{2(p)}h'_{1(p)} = \left(\frac{m^4}{C^2R^2} + \omega^2\right)\frac{ml}{2} \cos ml$$

$$= (m^4 + \omega^2C^2R^2)\frac{ml}{2C^2R^2} \cos ml$$

$$Y_{(p)} = Ep^2 \sin mx = E\frac{m^4}{C^2R^2} \sin mx$$

$$\therefore \frac{Y_{(p)}}{ph_{2(p)}h'_{1(p)}} = \frac{2Em^3 \sin mx}{l(\omega^2C^2R^2 + m^4)\cos ml}$$

thus

$$e_t = \sum_{s=1} \frac{2E(-1)^s s^3\pi^3 \sin \frac{s\pi x}{l}}{\omega^2C^2R^2l^4 + s^4\pi^4}\epsilon^{-\frac{s^2\pi^2}{CRl^2}t}$$

and

$$e = e_p + e_t$$

In using Duhamel's integral, it is again best to drop the terms pertaining to the permanent condition and to use the integral only for the transient term. It should be noted, however, that it is *not necessary* to do so. It is merely less laborious and the result obtained is in a familiar form.

The solution with unit function was shown to be

$$e = E\left[\frac{x}{l} + 2\sum_{s=1}^{s=\infty}(-1)^s \frac{\epsilon^{-\frac{s^2\pi^2}{CRl^2}t} \sin \frac{s\pi x}{l}}{s\pi}\right]$$

Note that the only term involving t is the exponential; everything else is constant as far as t is concerned.

$$\therefore \varphi(t) = k\epsilon^{-\alpha t} + \frac{x}{l}$$

$$\varphi(t-u) = k\epsilon^{-\alpha(t-u)} + \frac{x}{l}$$

$$\frac{d}{d(t-u)}\varphi(t-u) = -k\alpha\epsilon^{-\alpha(t-u)}$$

$$e(u) = E \cos \omega u$$

$$e(t) = E \cos \omega t$$

$$\phi(0) = \frac{x}{l} + 2\sum(-1)^s \frac{\sin \frac{s\pi x}{l}}{s\pi}$$

One part of the permanent term is, therefore,

$$E\phi(0) \cos \omega t$$

Referring to equation (45) (Chap. XIV), the second term becomes

$$e = \int_{u=0}^{u=t} -k\alpha\epsilon^{-\alpha(t-u)} E \cos \omega u \, du$$

$$= -k\alpha \frac{\cos(\omega t - \varphi) - \epsilon^{-\alpha t} \cos \varphi}{\sqrt{\alpha^2 + \omega^2}}, \text{ where } \tan \varphi = \frac{\omega}{\alpha}$$

Part of this expression belongs to the permanent term. The transient part is

$$e_t = \frac{Ek\alpha\epsilon^{-\alpha t} \cos \varphi}{\sqrt{\alpha^2 + \omega^2}}$$

$$= \frac{Ek\alpha^2}{(\alpha^2 + \omega^2)}\epsilon^{-\alpha t} = 2\sum_{s=1} \frac{E(-1)^s}{s\pi} \sin \frac{s\pi x}{l} \frac{s^4\pi^4}{C^2R^2l^4} \frac{C^2R^2l^4}{s^4\pi^4 + \omega^2C^2R^2l^4}\epsilon^{-\frac{s^2\pi^2 t}{CRl^2}}$$

which, by substituting, becomes

$$e_t = \sum_{s=1} \frac{2E(-1)^s s^3\pi^3 \sin \frac{s\pi x}{l}\, \epsilon^{-\frac{s^2\pi^2}{CRl^2}t}}{s^4\pi^4 + \omega^2C^2R^2l^4}$$

It may be of interest to determine, also, the value of the permanent alternating current under this condition. Referring to equation (57),

$$\underset{\cdot}{I} = \frac{E \cosh \underset{\cdot}{n}x}{\underset{\cdot}{z}_0 \sinh \underset{\cdot}{n}l} = \frac{E}{\underset{\cdot}{z}_0}\sqrt{\frac{\cosh^2 \alpha x - \sin^2 \alpha x}{\sinh^2 \alpha l + \sin^2 \alpha l}}\, /\underline{\varphi_3 - \varphi_2}$$

where $\tan \varphi_3 = \tanh \alpha x \cdot \tan \alpha x$. Since

$$\underset{\cdot}{z}_0 = \frac{\underset{\cdot}{z}}{\underset{\cdot}{n}} = \frac{R}{\sqrt{pCR}} = \sqrt{\frac{R}{\omega C}}\, /\underline{-45^\circ}$$

$$\underset{\cdot}{I} = E\sqrt{\frac{\omega C}{R}}\sqrt{\frac{\cosh^2 \alpha x - \sin^2 \alpha x}{\sinh^2 \alpha l + \sin^2 \alpha l}}\, /\underline{\varphi_3 - \varphi_2 + 45^\circ}$$

and the instantaneous value of the current is

$$i = E\sqrt{\frac{\omega C}{R}}\sqrt{\frac{\cosh^2 \alpha x - \sin^2 \alpha x}{\sinh^2 \alpha l + \sin^2 \alpha l}} \cos\left(\omega t + \varphi_3 - \varphi_2 + 45^\circ\right)$$

The current and voltage vectors are displaced at an angle $\varphi_2 - \varphi_3 - 45^\circ$.

Therefore, the power at any place if E is the maximum value of voltage at the generator is

$$P = \frac{EI}{2} \cos\left(\varphi_2 - \varphi_3 - 45^\circ\right)$$

$$= \frac{E^2}{2}\sqrt{\frac{\omega C}{R}}\sqrt{\frac{\cosh^2 \alpha x - \sin^2 \alpha x}{\sinh^2 \alpha l + \sin^2 \alpha l}} \cos\left(\varphi_2 - \varphi_3 - 45^\circ\right)$$

E is the maximum value of the voltage.

If the *"ideal" cable were very long*, then $\sinh \alpha l = \cosh \alpha l$ and both are very large compared with $\sin \alpha l$ and $\cos \alpha l$. The power given by the generator ($x = l$) is, then,

$$P_g = \frac{E^2}{2}\sqrt{\frac{\omega C}{R}} \cos 45^\circ = \frac{E^2}{2}\sqrt{\frac{\omega C}{2R}},$$

since $\tan \varphi_1 = \tan \varphi_3$.

The current at the generator would be $i = E\sqrt{\frac{\omega C}{R}} \cos(\omega t + 45^\circ)$.

CHAPTER XVII

TRANSMISSION LINE HAVING ALL FOUR CONSTANTS R, L, G, AND C

Let the problem be to find the current in a short-circuited line at any time and place after a storage battery has been connected to the circuit.

The general operational solution, as shown in Chap. XV, is

$$i = E\frac{Y}{n}\frac{\cosh nx}{\sinh nl}\mathbf{1} = E\frac{n}{Z}\frac{\cosh nx}{\sinh nl}\mathbf{1} \qquad (69)$$

$$= E\frac{m}{Z}\frac{\cos mx}{\sin ml}\mathbf{1} = E\frac{m \cos mx}{(R + pL)\sin ml}\mathbf{1} \qquad (70)$$

Since $Z_{(p)}$ is a product, $(R + pL)\sin ml$, one root will be $p_1 = -\frac{R}{L}$. The other will be a series of values corresponding to $\sin ml$ = zero, or $ml = \pm s\pi$, or $m = \pm\frac{s\pi}{l}$, beginning with $s = 1$, since the value corresponding to $p = 0$ is taken care of in the $Y_{(0)}/Z_{(0)}$ term.

The solution, then, will contain three sets of terms, the first corresponding to $Y_{(0)}/Z_{(0)}$, the second corresponding to the root $p_1 = -\frac{R}{L}$, and the third a series, or

$$i = E\left\{\frac{Y_{(0)}}{Z_{(0)}} + \frac{Y(p)}{ph_1'(p)h_2(p)}\epsilon^{p_1t} + \sum\frac{Y(p)}{ph_2'(p)h_1(p)}\epsilon^{p_2t}\right\}.$$

1. Find $Y_{(0)}/Z_{(0)}$.

Since

$$n^2 = (R + pL)(G + pC),\ n = \sqrt{RG} \text{ for } p = 0 \qquad (71)$$

and

$$Z_{(0)} = R \sinh nl = R \sinh l\sqrt{RG}$$

$$\therefore \frac{Y_{(0)}}{Z_{(0)}} = E\sqrt{\frac{G}{R}}\frac{\cosh x\sqrt{RG}}{\sinh l\sqrt{RG}} \qquad (72)$$

2. Find the second term. Let $h_1(p) = R + pL$ and $h_2(p) = \sin ml$

Then, referring to Chap. VI,

$$ph_1'(p)h_2(p) = pL \sin ml$$

Thus,

$$ph_1'(p)h_2(p), \text{ for } p = p_1 = -\frac{R}{L}, \text{ becomes } -R \sin ml$$

$$Y(p) = m \cos mx$$

but

$$\begin{aligned} -m^2 &= (R + pL)(G + pC) \\ &= p^2CL + p(RC + LG) + RG \end{aligned} \qquad (73)$$

If we substitute $p = p_1 = -\frac{R}{L}$, we get

$$-m_1^2 = \frac{R^2}{L^2}CL - \frac{R}{L}(RC + LG) + RG = 0 \quad \text{or } m_1 = 0$$

Thus, $Y(p)/ph_1'(p)h_2(p) = 0/0$. We must, therefore, find the value of this expression as m approaches zero. It is

$$\frac{Y(p)}{ph_1'(p)h_2(p)} = \lim_{m=0} \frac{m \cos mx}{-R \sin ml} = -\frac{1}{Rl}$$

Thus, the second term becomes

$$-\frac{E}{Rl}\epsilon^{-\frac{R}{L}t} \qquad (74)$$

3. Sin $ml = 0$ gives $m = \pm \frac{s\pi}{l}$ but

$$-m^2 = p^2CL + p(RC + LG) + RG$$

thus,

$$p = -\frac{RC + LG}{2CL} \pm \sqrt{\frac{(RC + LG)^2}{4C^2L^2} - \frac{RG + m^2}{CL}} = -\alpha \pm j\beta \qquad (75)$$

where

$$\alpha = +\frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right) \qquad (76)$$

and

$$\beta = \sqrt{\frac{m^2}{CL} - \frac{1}{4}\left(\frac{R}{L} - \frac{G}{C}\right)^2} \qquad (77)$$

α depends, as seen, upon the line constants only; β depends upon them and the particular terminal condition, also. In this case, the case of a short-circuited line, $\sin ml = 0$ gave $m = \pm \frac{s\pi}{l}$. Often, however, especially when the terminal condition is complicated, it is necessary to resort to graphs in order to find the root of m. This will be illustrated in Chap. XVIII.

We next proceed to find $ph_2'(p)h_1(p)$:

$$h_2'(p) = \frac{d}{dp}\sin ml = \frac{d}{dm}\sin ml\frac{dm}{dp} = l\cos ml\frac{dm}{dp} \qquad (78)$$

but

$$-m^2 = p^2CL + p(RC + LG) + RG$$

$$\therefore \frac{dm}{dp} = -\frac{2pCL + RC + LG}{2m}$$

$$\therefore h_2'(p) = -l\cos ml\frac{CL}{2m}\left[\left(\frac{R}{L} + \frac{G}{C}\right) + 2p\right]$$

$$\therefore ph_2'(p)h_1(p) = -pl\cos ml\,(R + pL)\frac{CL}{2m}\left[\left(\frac{R}{L} + \frac{G}{C}\right) + 2p\right]$$

thus,

$$\frac{Y(p)}{ph_2'(p)h_1(p)} = \frac{2Em^2\cos mx}{-pl\cos ml(R + pL)CL\left[\frac{R}{L} + \frac{G}{C} + 2p\right]}$$

By substituting $m^2 = \frac{s^2\pi^2}{l^2}$ and $p = -\alpha + j\beta$, one term results, which is multiplied by $\epsilon^{(-\alpha+j\beta)t}$. By substituting $p = -\alpha - j\beta$, another, similar term results which is multiplied by $\epsilon^{(-\alpha-j\beta)t}$. The two terms combine and the solution of the third term is

$$E\sum_{s=1}\frac{2(-1)^s l\cos\frac{s\pi x}{l}\epsilon^{-\alpha t}}{s^2\pi^2 + RGl^2}\left[\frac{1}{\beta}\left\{\frac{s^2\pi^2}{Ll^2} + \frac{1}{2}G\left(\frac{R}{L} - \frac{G}{C}\right)\right\}\sin\beta t - G\cos\beta t\right] \qquad (79)$$

The solution is the sum of the three terms given in equations (72), (74), and (79).

CHAPTER XVIII

TRANSMISSION LINE WITH IMPEDANCE AT THE SENDING AND RECEIVING ENDS

It has been shown that when the transmission line is either grounded or open, no difficulty exists in finding the roots of $Z_{(p)} = 0$. It often becomes quite difficult, however, if the terminal conditions are not so simple.

Heaviside discusses such cases in various parts of his book. Since, however, his treatment is somewhat different from that which will now be given, it will be well to illustrate one or two cases. Let us take, for instance, the case shown by Heaviside on page 143 (Vol. II), illustrated here in Fig. 33, in which the long horizontal lines may be the insulated conductor in a cable, whose outside sheath is grounded. z_1 and z_2 are concentrated impedances.

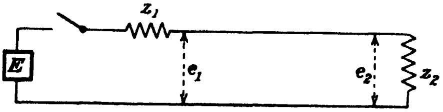

FIG. 33

From previous consideration, it is evident that as far as the transmission line itself is concerned, the following relations exist:

$$\left.\begin{aligned} e &= K_1 \cosh nx + K_2 \sinh nx \\ i &= \frac{Y}{n}[K_1 \sinh nx + K_2 \cosh nx] \end{aligned}\right\} \qquad (80)$$

For $x = 0$,

$$e = e_2 = i_2 z_2 = K_1(1) + 0 = K_1$$

for $x = l$,

$$e = e_1 = K_1 \cosh nl + K_2 \sinh nl$$

But

$$E = e_1 + i_1 z_1 \therefore E = i_2 z_2 \cosh nl + K_2 \sinh nl + i_1 z_1$$

For $x = 0$,

$$i_2 = \frac{Y}{n}[0 + K_2] \quad \therefore K_2 = \frac{ni_2}{Y}$$

For $x = l$,

$$i_1 = \frac{Y}{n}\left[i_2 z_2 \sinh nl + \frac{n}{Y} i_2 \cosh nl\right]$$

From these relations we obtain

$$E = i_2 z_2 \cosh nl + \frac{n}{Y} i_2 \sinh nl + \frac{Yz_1}{n} i_2\left[z_2 \sinh nl + \frac{n}{Y} \cosh nl\right]$$

$$= i_2\left[(z_1 + z_2) \cosh nl + \frac{1}{n}[Z + Yz_1z_2] \sinh nl\right]$$

$$= i_2 A \quad \therefore i_2 = \frac{E}{A}. \quad \text{Here, } Z = R + pL$$

From equation (80), $e = i_2 z_2 \cosh nx + \frac{ni_2}{Y} \sinh nx$. Thus,

$$e = \frac{z_2 \cosh nx + \frac{n}{Y} \sinh nx}{(z_1 + z_2) \cosh nl + \frac{1}{n}(Z + Yz_1z_2) \sinh nl} E\mathbf{1} \tag{81}$$

From equation (80), after substituting,

$$i = \frac{Y}{n} i_2\left[z_2 \sinh nx + \frac{n}{Y} \cosh nx\right]$$

$$= \frac{\frac{z_2}{z_0} \sinh nx + \cosh nx}{(z_1 + z_2) \cosh nl + \frac{1}{n}[Z + Yz_1z_2] \sinh nl} E\mathbf{1} \tag{82}$$

where

$$z_0 = \frac{n}{Y} = \frac{Z}{n}$$

Special Case if $z_1 = 0$ (*Fig.* 34), from equation (81)

$$e = \frac{z_2 \cosh nx + \frac{n}{Y} \sinh nx}{z_2 \cosh nl + \frac{Z}{n} \sinh nl} E\mathbf{1}$$

$$= \frac{z_2 \cosh nx + z_0 \sinh nx}{z_2 \cosh nl + z_0 \sinh nl} E\mathbf{1} \tag{83}$$

From equation (82), for $z_1 = 0$,

$$i = \frac{\frac{z_2}{z_0}\sinh nx + \cosh nx}{z_2\cosh nl + z_0\sinh nl}E1 \qquad (84)$$

At the receiving end $x = 0$.

$$\therefore e_2 = \frac{1}{\cosh nl + \frac{z_0}{z_2}\sinh nl}E1 \qquad (85)$$

and

$$i_2 = \frac{1}{z_2\cosh nl + z_0\sinh nl}E1 \qquad (86)$$

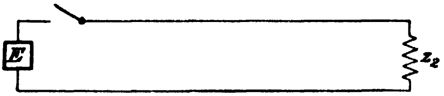

FIG. 34.

This current will be calculated in order to show the procedure when the roots $Z_{(p)} = 0$ cannot readily be obtained except by graphical means.

Assume again, for the sake of simplicity, that we deal with the "ideal" cable.

$$\therefore n^2 = pCR$$

If $m^2 = -n^2$, then $m^2 = -pCR$ and $p = -\frac{m^2}{CR}$

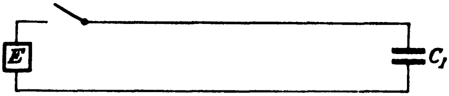

FIG 35.

Let the terminal impedance be a condenser of capacity C_1, as in Fig. 35.

$$\therefore z_2 = \frac{1}{pC_1} = -\frac{CR}{C_1m^2} \text{ and } z_0 = \frac{z}{n} = \frac{R}{\sqrt{pCR}} = \sqrt{\frac{R}{C}}\frac{\sqrt{CR}}{jm} = \frac{R}{jm}$$

Thus, since cosh jml = cos ml and sinh jml = j sin ml

$$i_2 = \frac{1}{-\frac{CR}{C_1 m^2}\cos ml + \frac{R}{m}\sin ml}E1$$

$$= \frac{\frac{C_1}{CR}m^2}{-\cos ml + \frac{C_1}{C}m\sin ml}E1$$

$$Z_{(p)} = 0 \text{ gives } \tan ml = \frac{C}{C_1 m}$$

What is wanted, then, are all possible values of m, m_1, m_2, m_3, etc., which satisfy the relation $\tan ml = C/C_1 m$

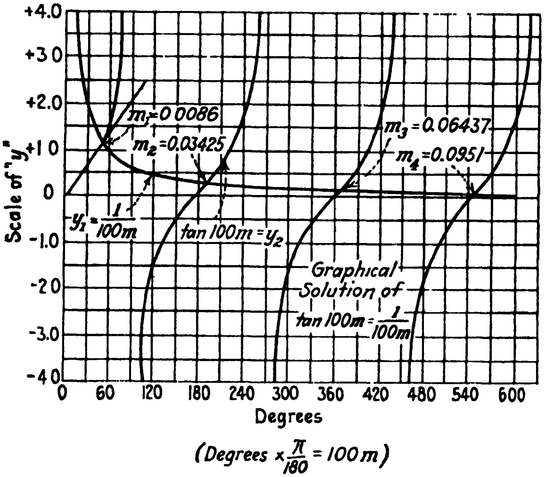

Fig 36

Figure 36 illustrates the method of obtaining these roots in a particular numerical case. An "ideal" cable (100 miles long) of resistance 88 ohms per mile, of capacity 0.054 microfarad per mile, is shunted at the distant end by a condenser C_1 of the same capacity as that of the entire cable. Find the equation for the current at the distant end when the cable is suddenly connected to a source of constant potential E.

The procedure used in finding the roots is to assign values to ml, then to plot the corresponding tangent curves and the graph C/C_1m.

Where these curves intersect, the roots are obviously located. These roots appear to be $m_1 = 0.0086$, $m_2 = 0.0343$, $m_3 = 0.0643$ etc. To find i_2, the current at the receiving circuit, we proceed in the usual way.

In this case (the "ideal" cable),

$$n^2 = (R + pL)(G + pC) = pCR \quad \therefore \quad -m^2 = pCR \qquad (87)$$

$$p\frac{dZ}{dp} = p\frac{dZ}{dm}\frac{dm}{dp}$$

$$\frac{dZ}{dm} = l \sin ml + \frac{C_1}{C}[ml \cos ml + \sin ml]$$

From equation (87),

$$\frac{dm}{dp} = -\frac{CR}{2m}$$

$$\therefore p\frac{dZ}{dp} = \frac{m}{2}\left(l + \frac{C_1}{C}\right)\sin ml + \frac{C_1}{2C}m^2 l \cos ml$$

$Y_{(0)}/Z_{(0)}$ corresponds to the steady case. Since we have a condenser at the end of the line, the final direct current is zero; thus, $Y_{(0)}/Z_{(0)} = 0$. This can be seen directly by making $p = 0$, which means $m = 0$.

The solution for the current at the receiving end of the line is

$$i = \sum_{m_1 m_2 m_3} \frac{\frac{C_1}{CR}m^2 \epsilon^{-\frac{m^2}{CR}t}}{\frac{m}{2}\left(l + \frac{C_1}{C}\right)\sin ml + \frac{C_1}{2C}m^2 l \cos ml}E$$

$$= \frac{2E}{Rl}\sum_{\alpha}\frac{\alpha \cos \alpha}{\alpha + \sin \alpha \cos \alpha}\epsilon^{-\frac{\alpha^2}{RCl^2}t}.$$

where $\alpha = ml$ and α is obtained from the equation $\tan \alpha = C/C_1 m = Cl/C_1\alpha$. Figure 37 gives the current at the end of the line for $E = 1$.

While Heaviside's operational calculus is primarily used to determine transients, it must not be forgotten that his operational solution is, also, most conveniently used to calculate the permanent alternating-current condition.

To illustrate this, assume a transmission line connecting a generator and a load. Assume, as is often the case, that the voltage and current at the receiving end are known. The problem is to find the voltage and current at the generator or at any point of the line. From equations (80),

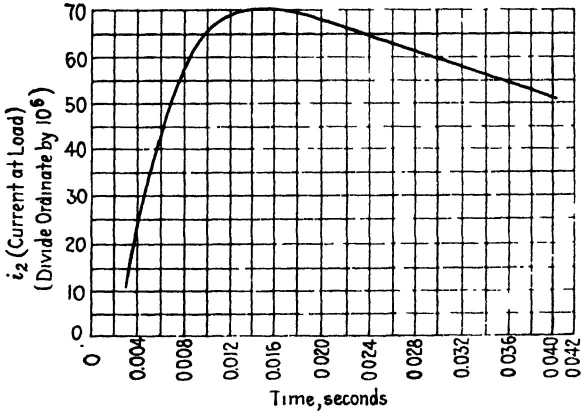

FIG. 37

$$e = K_1 \cosh nx + K_2 \sinh nx$$

$$i = \frac{Y}{n}[K_1 \sinh nx + K_2 \cosh nx]$$

for $x = 0$, $e = e_2$, and $i = i_2$, the known voltage and current at the receiving end. From this is seen that

$$K_1 = e_2 \text{ and } K_2 = \frac{ni_2}{Y}$$

$$\left.\begin{aligned} \therefore e &= e_2 \cosh nx + \frac{ni_2}{Y} \sinh nx \\ \text{and} \qquad & \\ i &= \frac{Y}{n} e_2 \sinh nx + i_2 \cosh nx \end{aligned}\right\} \qquad (87a)$$

If the transmission line is short—say, less than 5 per cent of the length of the alternating current wave—then a fair approximation to the true condition may be obtained by using only a few terms in the series representing sinh nx and cosh nx. That is, if with a 60-cycle system the transmission distance is less than

150 miles, the following approximation is satisfactory—indeed, the method may fairly well be used up to lines 200 miles long:

$$\cosh nx = 1 + \frac{n^2x^2}{2}$$

$$\sinh nx = nx$$

$$\therefore e = e_2\left[1 + \frac{n^2x^2}{2}\right] + \frac{ni_2}{Y}nx = e_2\left[1 + \frac{n^2x^2}{2}\right] + i_2 Zx$$

and

$$i = e_2\frac{Y}{n}\,nx + i_2\left[1 + \frac{n^2x^2}{2}\right] = e_2Yx + i_2\left[1 + \frac{n^2x^2}{2}\right]$$

Introducing vector notation

$$n^2 = YZ = (G + j\omega C)(R + j\omega L)$$

at the generator $x = l$

$$\therefore E_0 = e_2\left[1 + \frac{YZl^2}{2}\right] + I_2Zl \qquad \text{where } E_2 = e_2 + j\,0$$

$$I_0 = e_2Yl + I_2\left[1 + \frac{YZl^2}{2}\right]$$

If, however, the transmission line is long, this approximation is not sufficiently close and equations (87*a*) must be used. Fortunately, however, they are really quite simple.

Numerical Application.—Three-phase transmission line, 200 miles long, consisting of three No. 000 B.S. wires 10 feet apart. The voltage per phase at the receiving end of the line is 72,200 volts (corresponding to a line voltage of 125,000 volts). The particular value of the load current is taken as $I_2 = 100 - 50j$. Find the current and voltage at the generator.

The frequency is 60 cycles, and it is assumed that the leakage conductance is $0.192/10^6$ mho per mile. This corresponds to something like 1 kilowatt corona loss per mile of wire. Elementary considerations show that

R = resistance per mile = 0.326 ohm
L = inductance per mile = 0.00213 henry
C = capacity per mile = $0.0140/10^6$ farad
G = $0.192/10^6$ mho

$$\therefore Z = R + jX = 0.326 + j\,0.8035 = 0.866/\underline{67^\circ 53'}$$

$$Y = G + jB = \frac{0.192}{10^6} + j\frac{5.29}{10^6} = \frac{5.3}{10^6}/\underline{87^\circ 55'}$$

$$n = \sqrt{YZ} = \frac{2.14}{10^3}/77^\circ 54 = \frac{4.49}{10^4} + j\frac{20.95}{10^4}$$

From Chap. XXVIII,

$$\cosh nl = 0.9187\ /2^\circ 18' = 0.917 + j\,0.0367$$

$$\sinh nl = 0.418\ /\underline{78^\circ 39'} = 0.0823 + j\,0.41$$

$$I_2 = 100 - j50$$

Substituting these values in equation (87a) (when $x = l$), we find that

$$E_g = 81{,}850/10^\circ 49'$$

and

$$I_g = 100.5/18^\circ 28'$$

By "telescoping," we get the power delivered by one phase of the generator to be 8,140 kilowatts and the power delivered per phase, 7,220 kilowatts. In this case, the length of the line (200 miles) is rather too great for the use of the approximate equations. They give, as can readily be seen by substitution,

$$E_g = 81{,}800/11^\circ$$

$$I_g = 102.3/\underline{20^\circ}$$

Another example involving permanent alternating-current condition will be introduced because it involves some interesting and important features.

Referring to equations (83) and (84), it is seen that if $z_2 = z_0$ –that is, if the load impedance is the same as the surge impedance —then $i = e/z_0$ at any point of the line. The line behaves as an infinitely long line.

Problem.—Find the proper value of z_2, the load impedance at the end of an "ideal" cable. We have

$$z_0 = \frac{z}{n} = \frac{R}{\sqrt{pCR}}$$

Using vectors

$$z_0 = \frac{R}{\sqrt{j\omega CR}} = \frac{\sqrt{R}}{\sqrt{\omega C}/\underline{45^\circ}} = \sqrt{\frac{R}{\omega C}}/\underline{-45^\circ} = \sqrt{\frac{R}{2\omega C}} - j\sqrt{\frac{R}{2\omega C}}$$

$$= R_2 + jX_2$$

This shows that the end impedance should have a resistance $R_2 = \sqrt{\frac{R}{2\omega C}}$ ohms and a condensive reactance x_2, which is also $\sqrt{\frac{R}{2\omega C}}$ ohms.

The question may well be asked: Why write $\sqrt{j\omega CR} = \sqrt{\omega CR}\,\underline{/+45^\circ}$? Why not $\sqrt{\omega CR}\,/\pm 45^\circ$?

The reason for using the plus sign only is that the negative sign does not satisfy the condition that $p^{\frac{1}{2}}p^{\frac{1}{2}}\epsilon^{j\omega t}$ must be the same as $p\epsilon^{j\omega t}$, which is $j\omega\epsilon^{j\omega t}$. Thus,

$$p^{\frac{1}{2}}\epsilon^{j\omega t} = \sqrt{j\omega}\epsilon^{j\omega t} = \sqrt{\omega}\epsilon^{j\omega t}\,\underline{/45^\circ}$$

Therefore,

$$p^{\frac{1}{2}}p^{\frac{1}{2}}\epsilon^{j\omega t} = p^{\frac{1}{2}}(\sqrt{\omega}\epsilon^{j\omega t}/45^\circ = \omega\epsilon^{j\omega t}\underline{/45^\circ + 45^\circ} = j\omega\epsilon^{j\omega t}$$

This is the correct result. If we had used the negative angle, the result would obviously have been $p^{\frac{1}{2}}p^{\frac{1}{2}}\epsilon^{j\omega t} = \omega\epsilon^{j\omega t}\underline{/-45^\circ - 45^\circ} = -j\omega\epsilon^{j\omega t}$, which is wrong. Thus, we see that $\sqrt{j} = 1\underline{/45^\circ}$, not $1\underline{/-45^\circ}$

Referring again to equations (83) and (84), which are of special interest because they deal with a transmission line at the end of which is a certain load impedance:

In the process of solving them, one of the operations—and, indeed, the only laborious part—is to find the roots corresponding to $Z(p) = 0$.

We note that this gives

$$\tanh nl = -\frac{z_2}{z_0}$$

Since the majority of the readers are more familiar with trigonometric than hyperbolic functions, it is well to obtain the equivalent trigonometric expression.

Since $n^2 = -m^2, \qquad n = \pm jm$

Using the top sign at first, we get

$$\tanh(jml) = \frac{\sinh(jml)}{\cosh(jml)} = +j\tan ml$$

$$\because j\tan ml = -\frac{z_2}{z_0} = -\frac{z_2}{z}n = -\frac{z_2}{z}jm$$

$$\because \tan ml = -\frac{z_2}{z}m$$

Using the lower sign, we get the same relation,

$$\tanh(-jml) = \frac{\sinh(-jml)}{\cosh(-jml)} = -j\tan ml$$

$$\because -j\tan ml = -\frac{z_2}{z_0} = -\frac{z_2}{z}n = -\frac{z_2}{z}(-jm)$$

or

$$\tan ml = -\frac{z_2}{z}m \qquad \text{Q.E.D.}$$

In the case of an "ideal" cable, where $z = R$ and $-m^2 = pCR$, if the terminal load is a non-inductive resistance, then $z_2 = R_2$ and the relation becomes $\tan\alpha = -\frac{R_2}{Rl}\alpha$ where $\alpha = ml$.

If the load were a pure inductance L_2, then

$$\tan\alpha = \frac{L_2\alpha^3}{CR^2l^3}$$

If the load were a condenser, then

$$\tan\alpha = \frac{Cl}{C_2\alpha}$$

If the load impedance were $z_2 = R_2 + pL_2$, then

$$\tan\alpha = -\frac{R_2}{Rl}\alpha + \frac{L_2\alpha^3}{CR^2l^3}$$

If the load were a resistance in series with a condenser, then

$$\tan\alpha = -\frac{R_2}{Rl}\alpha + \frac{Cl}{C_2\alpha}$$

The numerical values of α which satisfy these relations can often be obtained best by plotting graphs, as shown in Fig. 36. The problem is, unfortunately, not always as simple as that illustrated. Complex roots frequently occur—then much labor is involved.

This whole subject of finding the roots is discussed in detail in Chap. XXVIII.

CHAPTER XIX

TWO OR MORE TRANSMISSION LINES OF DIFFERENT CONSTANTS CONNECTED TOGETHER BY AN IMPEDANCE OR SHUNTED AT THEIR JUNCTION BY AN IMPEDANCE

Consider, first, the case of a shunted impedance as shown in Fig. 38, in which case, for the sake of simplicity, it has been assumed that the line is short circuited at the receiving end.

The impedance or leak Z' is introduced at a point distant l_1 from the generator. Let the constants pertaining to the left- and right-hand sides have indices 1 and 2, respectively, and let the problem be to find the current at any point in either line.

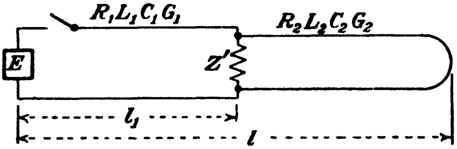

FIG. 38.

Consider, first, the right-hand side. The conditions existing there are those of a short-circuited transmission line discussed in Chap. XV. That is,

$$e = \frac{E' \sinh n_2 x}{\sinh n_2(l - l_1)}$$

and

$$i = \frac{Y_2}{n_2} E' \frac{\cosh n_2 x}{\sinh n_2 (l - l_1)}$$

where $l - l_1$ is the length of the right-hand line. Thus,

$$\frac{e}{i} = \frac{n_2 \sinh n_2 x}{Y_2 \cosh n_2 x}$$

At the leak, $x = l - l_1$

$$\therefore \quad z = \frac{e}{i} = \frac{n_2 \sinh n_2 (l - l_1)}{Y_2 \cosh n_2 (l - l_1)} \qquad (88)$$

We can, therefore, replace the right-hand short-circuited line by an impedance z placed at the junction. This impedance is in parallel with impedance Z'; thus, the joint impedance, which will be denoted by Z_2, is

$$Z_2 = \frac{zZ'}{z + Z'}$$

The current and voltage at any point on the left-hand transmission line of length l_1 can, therefore, be found from equations already worked out (see Chap. XVIII, equations (83) and (84)), where we deal with a line of constants having index 1, and where x is counted to the left from the junction point.

At the end of this line, which is the junction point in this problem, we find, from equation (85) (Chap. XVIII), that the voltage E^1 is

$$E^1 = \frac{1}{\cosh n_1l_1 + \frac{z_1}{n_1Z_2}\sinh n_1l_1}E1$$

Thus, for the right-hand line, the current and voltage are

$$e = \frac{1}{\cosh n_1l_1 + \frac{z_1}{n_1Z_2}\sinh n_1l_1}\,\frac{\sinh n_2x}{\sinh n_2(l - l_1)}E1 \qquad (89)$$

and

$$i = \frac{1}{\cosh n_1l_1 + \frac{z_1}{n_1Z_2}\sinh n_1l_1} \times \frac{Y_2}{n_2}\,\frac{\cosh n_2x}{\sinh n_2(l - l_1)}E1 \qquad (90)$$

Consider, two transmission lines of different constants (indices 1 and 2) connected together through an impedance Z', as shown in Fig. 39, where, again, for the sake of simplicity, the method is illustrated by the particular case of a short-circuited line.

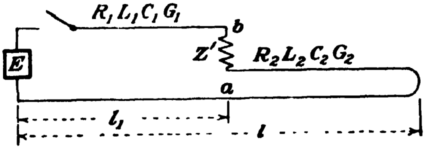

FIG. 39.

It is evident that up to point a, the impedance is z, as shown in equation (88). The impedance up to point b is, therefore, $Z_2 = Z' + z$.

Again, the values of the voltage and current for the left-hand side are obtained from equations (83) and (84) (Chap. XVIII), where we deal with a line of length l_1 and of constants having index 1.

At the end of this fictitious line represented in Fig. 40, the voltage and current can be obtained from equations (85) and (86) (Chap. XVIII).

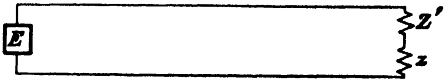

FIG. 40.

Let $Z' + z = Z_2$. Thus,

$$e_2 = \frac{1}{\cosh n_1 l_1 + \frac{z_1}{n_1 Z_2} \sinh n_1 l_1} E1$$

$$i_2 = \frac{1}{Z_2 \cosh n_1 l_1 + \frac{z_1}{n_1} \sinh n_1 l_1} E1$$

The voltage available across the second transmission line, the right-hand side of Fig. 39, is $e_3 = i_2 z$.

Thus, for the right-hand side, remembering that we now deal with a short-circuited line of length $l - l_1$,

$$e = \frac{z}{\cosh n_1 l_1 + \frac{z_1}{n_1 Z_2} \sinh n_1 l_1} \times \frac{\sinh n_2 x}{\sinh n_2 (l - l_1)} E1$$

and

$$i = \frac{z}{\cosh n_1 l_1 + \frac{z_1}{n_1 Z_2} \sinh n_1 l_1} \times \frac{Y_2}{n_2} \frac{\cosh n_2 x}{\sinh n_2 (l - l_1)} E1$$

Suppose that three transmission lines of different constants are connected together. Figure 41 illustrates such a case, a simple one, chosen purposely to illustrate the method to be used.

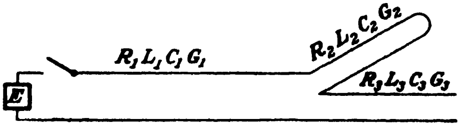

FIG. 41.

Branch 2 is short circuited, branch 3 open circuited. Find the current and voltage at any point of the network.

Each of the two branch lines is converted to impedances at the junction point. These are converted to one single impedance placed at the end of the main line, after which currents and voltages in the main line can be calculated from equations given in Chap. XVIII. Of special interest is the voltage at the junction point. This voltage having been obtained, the currents in the branch lines are readily calculated.

Thus, the equivalent impedance at the junction point of the open-circuited line is

$$z_3 = \frac{e}{i} = \frac{n_3 \cosh n_3 l_3}{Y_3 \sinh n_3 l_3} \text{ (see Chap. XV)}$$

and the equivalent impedance of the short-circuited line is

$$z_2 = \frac{n_2 \sinh n_2 l_2}{Y_2 \cosh n_2 l_2}$$

The combined impedance is $z = \frac{z_2 z_3}{(z_2 + z_3)}$

This is the impedance at the end of the main line and takes the place of z_2 in equations (83), (84), (85), and (86) of Chap. XVIII, and the results in the line are at once obtained; x is counted from the junction point toward the left.

Of special interest is e at the end of this fictitious line. It is

$$e = \frac{1}{\cosh n_1 l_1 + \frac{z_1}{z_2 n_1} \sinh n_1 l_1} E1$$

Therefore, the current and voltage in the open-circuited line are

$$e = \frac{1}{\cosh n_1 l_1 + \frac{z_1}{z_2 n_1} \sinh n_1 l_1} \times \frac{\cosh n_3 x}{\cosh n_3 l_3} E1$$

$$i = \frac{1}{\cosh n_1 l_1 + \frac{z_1}{z_2 n_1} \sinh n_1 l_1} \times \frac{Y_3 \sinh n_3 x}{n_3 \cosh n_3 l_3} E1$$

For the short-circuited branch,

$$e = \frac{1}{\cosh n_1 l_1 + \frac{z_1}{z_2 n_1} \sinh n_1 l_1} \times \frac{\sinh n_2 x}{\sinh n_2 l_2} E1$$

$$i = \frac{1}{\cosh n_1 l_1 + \frac{z_1}{z_2 n_1} \sinh n_1 l_1} \times \frac{Y_2 \cosh n_2 x}{n_2 \sinh n_2 l_2} E1$$

CHAPTER XX

A BATTERY CONNECTED AT SOME POINT OF A CABLE

In Fig. 42 is represented a cable having a grounded sheath. The left-hand end of the cable is grounded; the right-hand end is open circuited. It is desired to find the potential difference between the cable and the sheath after battery E has been inserted.

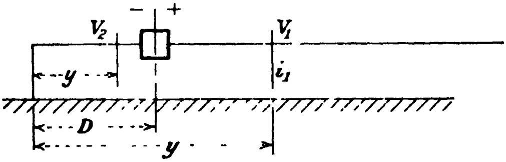

FIG. 42.

Referring to the general equations given in Chap. XV, and remembering that in them x is reckoned from the end of the line, we get for the right-hand side

$$V_1 = K_1 \cosh n\,(l - y) + K_2 \sinh n(l - y)$$
$$i_1 = \frac{Y}{n}[K_1 \sinh n(l - y) + K_2 \cosh n(l - y)]$$

For $y = l$, $i_1 = 0$; therefore, $K_2 = 0$

$$\therefore V_1 = K_1 \cosh n(l - y)$$

For the left-hand side,

$$V_2 = K_3 \cosh ny + K_4 \sinh ny$$

When $y = 0$, $V_2 = 0$; therefore, $K_3 = 0$

$$\therefore V_2 = K_4 \sinh ny$$

At $y = D$, the difference of potential between V_2 and V_1 must be the voltage of the battery. Therefore,

$$K_1 \cosh n(l - D) - K_4 \sinh nD = E$$

Since

$$\cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta = \cosh (\alpha + \beta)$$

it is seen by inspection that this condition exists if

$$K_1 = \frac{\cosh nD}{\cosh nl} E \text{ and } K_4 = -\frac{\sinh n(l-D)}{\cosh nl} E \text{ [1]}$$

Thus,

$$\left.\begin{aligned} V_1 &= \frac{\cosh nD}{\cosh nl} \cosh n(l-y)\, E\mathbf{1} \\ V_2 &= -\frac{\sinh n(l-D)}{\cosh nl} \sinh ny\, E\mathbf{1} \end{aligned}\right| \qquad (91)$$

Heaviside gives this solution in a slightly different form on page 140 (Vol. II). He uses m instead of n and trigonometric terms instead of hyperbolic. Since

$$m^2 = -n^2,\ n = jm$$

and since

$$\left.\begin{aligned} \sinh jm\alpha &= j \sin m\alpha \\ \cosh jm\alpha &= \cos m\alpha \end{aligned}\right\} \qquad (92)$$

$$\left.\begin{aligned} V_1 &= \frac{\cos mD \cos m(l-y)}{\cos ml} E\mathbf{1} \\ V_2 &= \frac{\sin m(l-D)}{\cos ml} \sin my\, E\mathbf{1} \end{aligned}\right| \qquad (93)$$

---

[1] These constants can also be obtained as follows:

$$V_1 = K_1 \cosh n(l-y), \quad i_1 = \frac{Y}{n} K_1 \sinh n(l-y)$$

$$V_2 = K_4 \sinh ny. \qquad i_2 = \frac{Y}{n} K_4 \cosh ny$$

For $y = D$, $V_1 - V_2 = E\mathbf{1}$ and $i_1 + i_2 = 0$

From these relations, K_1 and K_4 can be determined.

CHAPTER XXI

EFFECT OF ELECTRIC CHARGES APPLIED AT POINTS OF A TRANSMISSION LINE OR CABLE—GREEN'S FUNCTION

Figure 43 shows a point charge Q applied to a transmission wire. The problem is to find the potential difference between wire and ground after the charge has been applied.

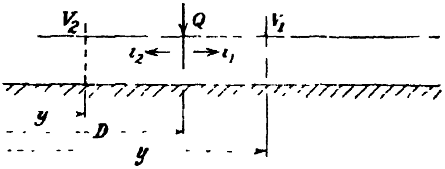

FIG. 43.

Since we are dealing with "distributed" constants, the general equations given in Chap. XV apply. For the right-hand side,

$$V_1 = K_1 \cosh n(l - y) + K_2 \sinh n(l - y)$$

$$i_1 = \frac{Y}{n}[K_1 \sinh n(l - y) + K_2 \cosh n(l - y)]$$

Since for

$$y = l,\ i_1 = 0 \text{ and, therefore, } K_2 = 0$$

$$\therefore V_1 = K_1 \cosh n(l - y)$$

and

$$i_1 = \frac{Y}{n} K_1 \sinh n(l - y)$$

For the left-hand side,

$$V_2 = K_3 \cosh ny + K_4 \sinh ny$$

$$i_2 = \frac{Y}{n}[K_3 \sinh ny + K_4 \cosh ny]$$

For

$$y = 0,\ i_2 = 0 \text{ and, therefore, } K_4 = 0$$

$$\therefore V_2 = K_3 \cosh ny$$

and

$$i_2 = \frac{Y}{n} K_3 \sinh ny$$

The next step is to connect these currents with the applied charge.

The problem involves placing an instantaneous charge at $y = D$ at $t = 0$. The rate at which the charge enters the line element is dQ/dt, which is a current.

In this case, as will be evident from the discussion which follows, it is $d/dt\ (Q1)$ or $Qp1$ where Q is the numerical value of the applied charge.

If the current is written as $Qp1$, the charge given to the element during a short time interval is

$$\int_{t-\epsilon}^{t+\epsilon} Qp1dt = Q\int_{-\epsilon}^{t+\epsilon} \frac{d}{dt}1dt = Q\Big[1\Big]_{t-\epsilon}^{t+\epsilon}$$

If t is not $= 0$, this becomes

$$Q[1 - 1] = 0$$

If $t = 0$, it becomes

$$Q[1 - 0] = Q$$

Thus, by writing $i = i_1 + i_2 = Qp1$ we satisfy the imposed condition, namely, that the element receives the full charge at $t = 0$ and no charge at all at any other time. Since

$$i_1 + i_2 = pQ1,$$

$$\frac{Y}{n}[K_1 \sinh n(l - D) + K_3 \sinh nD] = pQ1$$

This is satisfied if

$$K_1 = \frac{\cosh nD}{\sinh nl} pQ \frac{n}{Y} 1$$

and

$$K_3 = \frac{\cosh n(l - D)}{\sinh nl} pQ \frac{n}{Y} 1$$

$$\therefore V_1 = \frac{\cosh nD}{\sinh nl} \cosh n(l - y) \frac{npQ}{Y} 1 \tag{94}$$

$$V_2 = \frac{\cosh n(l - D)}{\sinh nl} \cosh ny \frac{npQ}{Y} 1 \tag{95}$$

In the case of a cable without leakage and without inductance, $Y = pC$, and the equations become

$$V_1 = \frac{\cosh nD}{\sinh nl} \cosh n(l - y) \frac{nQ}{C} 1 \qquad (96)$$

$$V_2 = \frac{\cosh n(l - D)}{\sinh nl} \cosh ny \frac{nQ}{C} 1 \qquad (97)$$

Heaviside gives a large number of similar problems, all of which can readily be solved in a manner similar to that just indicated.

It may be instructive to solve one of these equations. Find, for instance, V_1, the potential on the right-hand side in the case of an "ideal" cable.

Equation (96), rewritten with trigonometric relations, is

$$V_1 = \frac{\cos mD \cos m(l - y)}{j \sin ml} \frac{jm\,Q}{C} 1, \text{ since } -m^2 = pCR$$

$$= \frac{m \cos mD \cos m(l - y)}{C \sin ml} Q1$$

$$Z_{(p)} = 0 \text{ gives roots } ml = \pm s\pi \qquad \text{or } m = \pm\frac{s\pi}{l}$$

also,

$$-m^2 = pCR$$

Thus,

$$\frac{pdZ}{dp} = -\frac{s^2\pi^2}{CRl^2} \frac{l^2 CR}{2s\pi} \cos ml = (-1)^s \frac{s\pi}{2}$$

$$Y_{(p)} = (-1)^s \cos mD \cos my \frac{s\pi}{l}, \frac{Y_{(0)}}{Z_{(0)}} = \frac{Q}{Cl}$$

$$\therefore V_1 = \frac{Q}{Cl}\left[1 + 2\sum_{s=1}^{s=\infty} \cos my \cos mD\, \epsilon^{-\frac{s^2\pi^2}{CRl^2}t}\right]$$

The equation for V_2 becomes identical with that for V_1, as can be seen from equations (96) and (97). They are alike if we interchange y and D. The values of V_1 and V_2 differ, however, numerically, since, in the first case, y is always larger than D, and in the second, it is smaller. Thus, in this case,

$$V = \frac{Q}{Cl}\left[1 + 2\sum_{s=1}^{s=\infty} \cos\frac{s\pi y}{l} \cos\frac{s\pi D}{l}\, \epsilon^{-\frac{s^2\pi^2}{CRl^2}t}\right] \qquad (98)$$

This equation holds for both sides.

For certain reasons, it is well to study two more cases where a charge is suddenly applied to a point in a transmission line wire.

For this purpose, we shall consider a wire which is grounded at both ends, as in Fig. 44. Here, the procedure is similar to that which has been described, and the following operational equations are readily obtained:

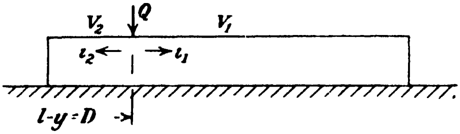

FIG. 44.

$$V_1 = \frac{\sinh nD \sinh n(l-y)}{\sinh nl}\frac{npQ}{Y}1 \tag{99}$$

$$V_2 = \frac{\sinh n(l-D)\sinh ny}{\sinh nl}\frac{npQ}{Y}1 \tag{100}$$

When these equations are solved for the conditions of an "ideal" cable by the expansion theorem, we find

$$V_1 = \frac{2Q}{Cl}\sum_{s=1}^{s=\infty}\sin\frac{s\pi y}{l}\sin\frac{s\pi D}{l}\epsilon^{-\frac{s^2\pi^2}{CRl^2}t} \tag{101}$$

and V_2 is identical with V_1, as far as the general equation is concerned. Obviously, however, the values of y in the first case are larger than D, and in the second case are smaller.

In this case, then,

$$V = \frac{2Q}{Cl}\sum_{s=1}^{s=\infty}\sin\frac{s\pi y}{l}\sin\frac{s\pi D}{l}\epsilon^{-\frac{s^2\pi^2}{CRl^2}t} \tag{102}$$

This relation holds for the entire line.

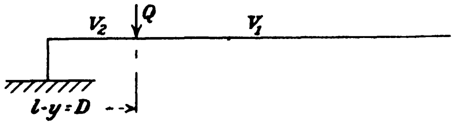

FIG. 45.

A third similar case is shown in Fig. 45. The left-hand side is here grounded and the right-hand side open circuited. Here the operational equations become

$$V_1 = \frac{\sinh nD \cosh n(l-y)}{\cosh nl} \frac{pnQ}{Y} \mathbf{1}$$

$$V_2 = \frac{\sinh ny \cosh n(l-D)}{\cosh nl} \frac{pnQ}{Y} \mathbf{1}$$

The two expressions are identical if y and D are interchanged.

The solution will again be the same for both sides, so that the potential at any point is

$$V = \frac{2Q}{Cl} \sum_{s=1}^{s=\infty} \sin\left(\frac{2s-1}{2}\frac{\pi}{l}D\right) \sin\left(\frac{2s-1}{2}\frac{\pi}{l}y\right) \epsilon^{-\frac{(2s-1)^2}{4}\frac{\pi^2}{CRl^2}t} \quad (103)$$

In all three cases, there is no potential on the line at $t = 0$ except at the point $y = D$, where it is infinity, because a finite charge is given to a point of infinitely small capacity.

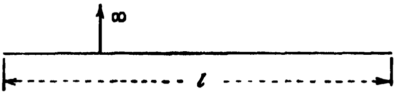

FIG. 46.

A graph indicating the *physical significance* of these equations when they are used to express arbitrary functions is given in Fig. 46.[1] They are very useful when, for instance, we are dealing with space distributions of charges, and they are similar to Duhamel's integral when time distributions are involved. It is to be noted, therefore, that these equations represent an impulsive function similar to $p\mathbf{1}$, with the difference that the variable is y instead of t.

When we integrate $p\mathbf{1}$, we get back to the unit function, that is, $\frac{1}{p} p\mathbf{1} = \mathbf{1}$. So here the space total is $\frac{Q}{C}\mathbf{1}$.

The three equations, (98), (102), and (103), when Q/C is taken as unity, are special types of Green's functions, which are exceedingly useful to express arbitrary functions in trigonometric series.[1] We shall denote them by G. Thus,

---

[1] Although we have shown these functions in Fig. 46 as straight lines which are zero everywhere except where $y = D$, they are, in fact, infinitely oscillating over the whole range, and the oscillations reach infinity at $y = D$.

$$G = \frac{1}{l}\left[1 + 2\sum_{s=1}^{s=\infty} \cos\frac{s\pi}{l}y \cos\frac{s\pi}{l}D\right]$$

or

$$= \frac{2}{l}\sum_{s=1}^{s=\infty} \sin\frac{s\pi y}{l}\sin\frac{s\pi D}{l}$$

or

$$= \frac{2}{l}\sum_{s=1}^{s=\infty} \sin\frac{2s-1}{2}\frac{\pi}{l}y \sin\frac{2s-1}{2}\frac{\pi}{l}D$$

Every Fourier's series involves one form of these. An infinite number can be worked out by assuming different terminal arrangements.

Each Green's function is an impulsive function.

If we multiply a continuous function of D—say, $f(D)$—by a Green's function, which is an impulsive function that exists only at $D = y$, the product is zero except at that particular point, where it is infinite. But, if we take the space total of the product $Gf(D)$, the result is $f(y)$ because G exists only at $D = y$ and its total is 1.

Thus, $\int Gf(D)dD = f(y)$ if the limits include the point $D = y$. If not, the result is zero.

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[1] Heaviside gives a most interesting discussion on this subject in his treatise on "Electromagnetic Theory," Vol. II, p. 99.

An analogue between pure mathematics and operational mathematics of Heaviside by means of the theory of H functions was prepared by Dr. J. J. Smith and appeared in *Jour. Franklin Inst.*, November, 1925.

CHAPTER XXII

DISCONNECTING AND DISCHARGING A CABLE

DISCONNECTING A CABLE

Before the switch is opened, the current, in this example (Fig. 47), is assumed to have reached its steady state, that is, $I = \frac{E}{Rl}$ in all parts of the cable. After the switch is opened, the current at the switch is zero.

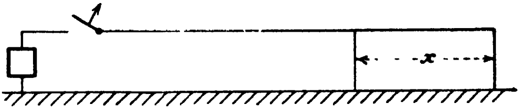

FIG. 47.

If, therefore, we determine the expression for some peculiar e.m.f. impressed at the generator, such that at the generator the current is constant, equaling $-I$ at all times after closing the switch, then the sum of the original current and the current caused by this peculiar e.m.f. will obviously be the current in any part of the cable.

It has been shown that with the receiving end of a cable grounded,

$$e = e_0 \frac{\sinh nx}{\sinh nl}$$

and

$$i = \frac{Y}{n} e_0 \frac{\cosh nx}{\sinh nl}$$

$$\therefore i_0 = e_0 \frac{Y}{n} \frac{\cosh nl}{\sinh nl}$$

The peculiar e.m.f. which makes $i_0 = -I1$ is, then,

$$e_0 = -\frac{n \sinh nl}{Y \cosh nl} I1$$

and thus the current in any part of the cable due to this e.m.f. is

$$i_1 = -\frac{Y\cosh nx}{n\sinh nl}\,\frac{n\sinh nl}{Y\cosh nl}I\mathbf{1} = -\frac{\cosh nx}{\cosh nl}I\mathbf{1} \tag{104}$$

The total current is, then,

$$i = i_1 + I = I\left[1 - \frac{\cosh nx}{\cosh nl}\mathbf{1}\right]$$

$$= \frac{E}{Rl}\left[1 - \frac{\cosh nx}{\cosh nl}\mathbf{1}\right]$$

At the end of the line, $x = 0$

$$\therefore i_r = \frac{E}{Rl}\left[1 - \frac{1}{\cosh nl}\mathbf{1}\right]$$

$$i_1 = -\frac{I}{\cosh nl}\mathbf{1} = -\frac{Y_{(p)}I}{Z_{(p)}}\mathbf{1}$$

Problem.—Find the current at the grounded end of an "ideal" cable.

Refer to equation (104) for $x = 0$.

$$i_1 = -\frac{I}{\cosh nl}\mathbf{1}$$

$$Y_{(p)} = 1$$

$$Z_{(p)} = \cosh nl = \cos ml$$

where

$$-m^2 = n^2 = pCR$$

$$p\frac{dZ}{dp} = p\frac{dZ}{dm}\frac{dm}{dp} = -\frac{ml}{2}\sin ml$$

$$\frac{Y_{(o)}}{Z_{(o)}} = \frac{1}{\cos 0} = 1$$

$$\therefore i_1 = -\left[I + \sum \frac{2I\epsilon^{pt}}{-ml\sin ml}\right]$$

$$Z_{(p)} = 0 \text{ gives } \cos ml = 0 \quad \therefore ml = \frac{\pi}{2}, \frac{3\pi}{2}, \dots \pm\frac{k\pi}{2}$$

or

$$ml = \pm\frac{2s+1}{2}\pi$$

$$\therefore p = -\left(\frac{2s+1}{2}\right)^2\frac{\pi^2}{CRl^2}$$

$$\therefore i_1 = -\left[I - \sum_{s=0}^{s=\infty}\frac{2I\epsilon^{pt}}{\frac{2s+1}{2}\pi\sin\frac{2s+1}{2}\pi}\right]$$

$$\therefore i = I + i_1 = \frac{4E}{Rl\pi}\sum_{s=0}^{s=\infty}\frac{\cos s\pi}{2s+1}\epsilon^{-\alpha t} \tag{105}$$

where

$$\alpha = \left(\frac{2s+1}{2}\right)^2 \frac{\pi^2}{CRl^2}$$

At $t = 0$, the current is E/Rl. Equation (105) gives

$$i = \frac{4E}{Rl\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\right]$$

Thus, $\frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} \cdots\right]$ must be unity, a relation which is well known.

DISCHARGING A CHARGED CABLE

Referring now to Fig. 48, switch a is closed from $t = 0$ to $t = t_1$, at which time switch b is closed. Find the current in the cable. Short circuiting the battery means that the impressed e.m.f. is zero at time t_1. This, again, is equivalent to leaving the original battery on but applying a battery of potential $-E\mathbf{1}$ at time $t = t_1$. The total current is, then, the sum of the currents due to these two e.m.fs.

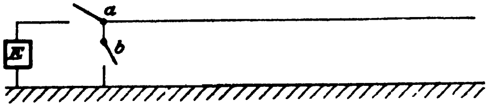

FIG. 48

Current due to $E\mathbf{1}$ is $i_1 = \frac{Y}{n}E\frac{\sinh nx}{\cosh nl}\mathbf{1} = E\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1}$, as is shown in equation (56).

This is readily solved by the expansion theorem, which gives

$$i_1 = \frac{2E}{Rl}\sum_0^\infty (-1)^s \sin\left(\frac{2s+1}{2}\pi\frac{x}{l}\right)\epsilon^{\beta t}$$

When e.m.f. $-E\mathbf{1}$ is applied at $t = t_1$, the current is obviously

$$i_2 = -\frac{2E}{Rl}\sum_0^\infty (-1)^s \sin\left(\frac{2s+1}{2}\pi\frac{x}{l}\right)\epsilon^{\beta(t-t_1)}$$

$$\therefore i = \frac{2E}{Rl}\left[\sum_0^\infty (-1)^s \sin\left(\frac{2s+1}{2}\pi\frac{x}{l}\right)\epsilon^{\beta t} - \sum_0^\infty (-1)^s \sin\left(\frac{2s+1}{2}\pi\frac{x}{l}\right)\epsilon^{\beta(t-t_1)}\right]$$

where

$$\beta = \left(\frac{2s + 1}{2}\right)^2 \frac{\pi^2}{RCl^2}$$

The corresponding solution, if the cable should be connected to earth at the receiving end, is easily found to be

$$i = \frac{2E}{Rl}\left[\sum_1^\infty (-1)^s \cos\left(s\pi\frac{x}{l}\right)\epsilon^{\alpha t} - \sum_1^\infty (-1)^s \cos\left(s\pi\frac{x}{l}\right)\epsilon^{\alpha(t-t_1)}\right]$$

where

$$\alpha = \frac{s^2\pi^2}{CRl^2}$$

A similar problem has been worked out at the end of Chap. XV.

CHAPTER XXIII

FRACTIONAL DIFFERENTIATION AND INTEGRATION

The second volume of Heaviside's "Electromagnetic Theory' contains a great deal of suggestive matter in reference to the interpretation of divergent series and fractional derivatives. Much of it is very difficult to understand, and, indeed, some of his results are given with a statement that "this is not so simple as it looks."

Heaviside illustrates his use of operational methods by many examples, and the final solutions are always correct, as far as the writer has checked them. But the reasons suggested are not always clear, often they are far from clear.

The difficulties arise in connection with the asymptotic solution when the operational equation contains fractional powers of p.

In some problems, the series obtained by division in rising powers of p, gives the complete solution; in others, certain terms have to be added. This matter is discussed in Chap. XXV.

Heaviside arrived at the following relation:

$$p^{\frac{1}{2}}\mathbf{1} = \frac{1}{\sqrt{\pi t}}\mathbf{1} \qquad (106)$$

from two different points of view.

1. From the solution of certain heat problems worked out by Fourier, which will be shown later.

2. By generalizing the formula for the derivatives of any power of t.

It is

$$\frac{d}{dt}\left(\frac{t^n}{\underline{|n}}\right) = \frac{t^{n-1}}{\underline{|n-1}} = p^1\frac{t^n}{\underline{|n}} = \frac{t^{n-1}}{\underline{|n-1}}$$

or, more generally,

$$p^m\frac{t^n}{\underline{|n}}\mathbf{1} = \frac{t^{n-m}}{\underline{|n-m}}\mathbf{1} \qquad (107)$$

There is no difficulty about this as long as m and n are positive integers and $n \geqq m$. When $m > n$, we get the factorial of a negative integer, which is minus infinity, and the result is zero.

Heaviside felt that equation (107) in his system should hold even when m and n are fractions and proceeded to study them.

The unit function would then correspond to $t^0/\underline{|0}$, or we would write $\frac{t^0}{\underline{|0}}\boldsymbol{1}$. This is rather remarkable, in a way, since $t^n/\underline{|n}$ as n approaches zero is a peculiar graph. It is a cylindrical surface for all positive and negative values of t, a surface which collapses to a point at $t = 0$. Therefore, the positive real part would look like $\boldsymbol{1}$, the "unit function."

But to return to $p^{\frac{1}{2}}\boldsymbol{1} = \frac{1}{\sqrt{\pi t}}$ as Heaviside writes it, or $p^{\frac{1}{2}}\boldsymbol{1} = \frac{1}{\sqrt{\pi t}}\boldsymbol{1}$ as we prefer to write it.

If the graph corresponding to $1/\sqrt{\pi t}$ is plotted for various values of t, we find that it starts at infinity and decreases to zero. When this chart is multiplied by the unit function, it is in no way modified. Thus, $p^n\left(\frac{1}{\sqrt{\pi t}}\boldsymbol{1}\right)$ is obtained by differentiating $1/\sqrt{\pi t}$ and multiplying the result by the unit function.

The question remains, however, whether equation (106) is true. It can be tested in various ways. We may, for instance, see whether

$$p^{\frac{1}{2}}(p^{\frac{1}{2}}\boldsymbol{1}) = p\boldsymbol{1} \tag{108}$$

or

$$p^{-\frac{1}{2}}(p^{\frac{1}{2}}\boldsymbol{1}) = \boldsymbol{1} \tag{109}$$

or

$$p^{\frac{1}{2}}(p^{\frac{1}{2}} \sin \omega t\boldsymbol{1}) = \omega \cos \omega t\boldsymbol{1}, \text{ etc.} \tag{110}$$

Consider equation (106) and refer to equation (107). If $m = \frac{1}{2}$ and $n = 0$, we get $= p^{\frac{1}{2}}\frac{t^0}{\underline{|0}} = \frac{t^{-\frac{1}{2}}}{\underline{|-\frac{1}{2}}}$. But $\underline{|-\frac{1}{2}} = \sqrt{\pi}$, as will be shown later.

Thus, $p^{\frac12}\frac{t^0}{\underline{|0}} = \frac{1}{\sqrt{\pi t}}$. This is certainly similar to equation (106) and would lead us to suspect that

$$p^{\frac12}\left(\frac{t^0}{\underline{|0}}\mathit{1}\right) = \frac{1}{\sqrt{\pi t}}\mathit{1} \qquad (111)$$

Try, now, to verify equation (108) by the use of equations (107) and (111). The left-hand side gives

$$p^{\frac12}p^{\frac12}\left(\frac{t^0}{\underline{|0}}\mathit{1}\right) = p\left(\frac{t^0}{\underline{|0}}\mathit{1}\right)$$

The right-hand side gives

$$p^{\frac12}\left(\frac{t^{-\frac12}}{\underline{|-\frac12}}\mathit{1}\right) = \frac{t^{-1}}{\underline{|-1}}\mathit{1}$$

Thus the unit function could well be written and in its most completed form should be written $\frac{t^0}{\underline{|0}}\mathit{1}$ and $p\mathit{1} = \frac{t^{-1}}{\underline{|-1}}\mathit{1}$

Consider, next, equation (109).

$$p^{-\frac12}p^{\frac12}\left(\frac{t^0}{\underline{|0}}\mathit{1}\right) = p^{-\frac12}\frac{t^{-\frac12}}{\underline{|-\frac12}}\mathit{1}$$

The left-hand side gives $\frac{t^0}{\underline{|0}}\mathit{1}$. The right-hand side gives, also, $\frac{t^0}{0}\mathit{1}$, when we again consider the unit function as a multiplying factor.

Finally, consider equation (110).

$$p^{\frac12}\sin\omega t\mathit{1} = p^{\frac12}\left(\omega t - \frac{\omega^3t^3}{\underline{|3}} + \cdots\right)\mathit{1}$$

$$= \left(\omega\frac{t^{\frac12}}{\underline{|1\frac12}} - \frac{\omega^3t^{\frac52}}{\underline{|5\frac12}} + \cdots\right)\mathit{1}.$$

This can be shown to be a converging series and can also be written in the following more convenient form, which will be treated in Chap. XXVI:

$$p^{\frac12}(p^{\frac12}\sin\omega t\mathit{1}) = \left(\frac{\omega t^0}{\underline{|0}} - \frac{\omega^3t^2}{\underline{|2}} + \cdots\right)\mathit{1} = \omega\cos\omega t\mathit{1}$$

It is seen that fractional differentiation and integration introduce factorials of fractions.

The so-called *π function* (pi function) is the generalized factorial. Its value can be obtained from the gamma function or directly by series expansion:

$$\pi(x) = \Gamma(x+1) = \lim_{n=\infty} \frac{1\cdot 2\cdot 3 \cdots (n-1)n}{(x+1)(x+2)\cdots(x+n)} n^x \quad (112)$$

The most important values for our purposes are the following:

x	0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\pi(x) = \underline{\lvert x} =$	1	$\frac{\sqrt{\pi}\,1}{2}$	$\frac{\sqrt{\pi}\,1\,3}{2\,2}$	$\frac{\sqrt{\pi}\,1\,3\,5}{2\,2\cdot 2}$

x	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$
$\pi(x) = \underline{\lvert x} =$	1	$+\sqrt{\pi}$	$\frac{-2\sqrt{\pi}}{1}$	$\frac{+2\,2\sqrt{\pi}}{1\cdot 3}$	$\frac{-2\cdot 2\,2\sqrt{\pi}}{1\,3\,5}$

Since a number of problems involve fractional derivatives of the unit function, the following tabulation will prove a convenience:

$$\left.\begin{aligned}
p^{\frac{1}{2}}1 &= \frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}1 & p^{-\frac{1}{2}}1 &= \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}}1 \\
p^{\frac{3}{2}}1 &= \frac{-t^{-\frac{3}{2}}}{2\sqrt{\pi}}1 & p^{-\frac{3}{2}}1 &= \frac{2\cdot 2t^{\frac{3}{2}}}{1\cdot 3\sqrt{\pi}}1 \\
p^{\frac{5}{2}}1 &= \frac{1\cdot 3t^{-\frac{5}{2}}}{2\cdot 2\sqrt{\pi}}1 & p^{-\frac{5}{2}}1 &= \frac{2\cdot 2\cdot 2t^{\frac{5}{2}}}{1\cdot 3\cdot 5\sqrt{\pi}}1 \\
p^{\frac{7}{2}}1 &= \frac{-1\cdot 3\cdot 5t^{-\frac{7}{2}}}{2\cdot 2\cdot 2\sqrt{\pi}}1 & p^{-\frac{7}{2}}1 &= \frac{2\cdot 2\cdot 2\cdot 2t^{\frac{7}{2}}}{1\cdot 3\cdot 5\cdot 7\sqrt{\pi}}1
\end{aligned}\right\} \quad (112a)$$

Before leaving this interesting subject, it may be well to call attention to another feature connected with fractional differentiation.

Take, for instance, successive derivatives of $t/\underline{|1}$.

$$\frac{d}{dt}\frac{t}{\underline{|1}} = \frac{t^0}{\underline{|0}}$$

$$\frac{d^2}{dt^2}\frac{t}{\underline{|1}} = \frac{t^{-1}}{\underline{|-1}}$$

$$\frac{d^3}{dt^3}\frac{t}{\underline{|1}} = \frac{t^{-2}}{\underline{|-2}}$$

$$\frac{d^n}{dt^n}\frac{t}{\underline{|1}} = \frac{t^{-n}}{\underline{|-n}}$$

The last three expressions are usually written as zero, since the factorial of a negative integer is ∞. When operated upon with $p^{\frac{1}{2}}$, however, they give $\frac{t^{-1\frac{1}{2}}}{\underline{|-1\frac{1}{2}}}$, $\frac{t^{-2\frac{1}{2}}}{\underline{|-2\frac{1}{2}}}$, and $\frac{t^{-n-\frac{1}{2}}}{\underline{|-n-\frac{1}{2}}}$, respectively. All three are definite and different. There are zeros and zeros.

CHAPTER XXIV

INDEFINITELY LONG TRANSMISSION LINE OR CABLE

Consider a transmission line that is of infinite length or, at any rate, so long that whether the line is open or closed at the far end no potential difference exists at the far end. This is a situation which very nearly exists in very long telegraph cables.

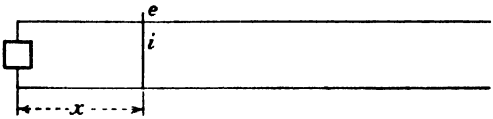

FIG. 49.

In these problems, it is obviously best to count the distance x from the sending end rather than from the receiving end, as was done in Chap. XV.

Referring to Fig. 49,

$$\frac{\partial e}{\partial x} = -i(R + pL)$$

$$\frac{\partial i}{\partial x} = -e(G + pC)$$

Thus,

$$\frac{\partial^2 e}{\partial x^2} = YZe = n^2 e$$

$$\therefore e = A\epsilon^{nx} + B\epsilon^{-nx}$$

For $x = \infty$, $e = 0$; therefore, $A = 0$ and $e = B\epsilon^{-nx}$. For $x = 0$, $e = E$; therefore, $B = E$ and

$$e = E\epsilon^{-nx}\mathbf{1} \tag{113}$$

$$i = \frac{Y}{n}E\epsilon^{-nx}\mathbf{1} \tag{114}$$

Therefore, e/i, the resistance operator or unit function, is n/Y at any point of the line.

In the case of an "ideal" cable, $Y = pC$, $Z = R$, and

$$\frac{n}{Y} = \frac{\sqrt{pCR}}{pC} = \left(\frac{R}{pC}\right)^{\frac{1}{2}} \qquad (115)$$

The current at the battery is

$$i_0 = \left(\frac{pC}{R}\right)^{\frac{1}{2}} E\mathbf{1}$$

and

$$\frac{i}{i_0} = \frac{\frac{Y}{n}E\epsilon^{-nx}}{\frac{Y}{n}E} = \epsilon^{-nx}$$

Thus,

$$i = E\epsilon^{-nx}\left(\frac{pC}{R}\right)^{\frac{1}{2}}\mathbf{1} \qquad (116)$$

A similar problem in heat conduction through an infinitely long bar had many years previously been worked out with conventional mathematics involving Fourier's series. The differential equation relating to the temperature is

$$\frac{d\theta}{dt} = h^2\frac{d^2\theta}{dx^2}. \qquad (117)$$

where h^2 is termed by Kelvin the *diffusivity* and by Maxwell the *thermometric conductivity.*

The rate of heat transfer through unit area is

$$W = -K\frac{d\theta}{dx} \qquad (118)$$

where K is the thermal conductivity of the bar.

When these equations are solved, subject to the condition that at the origin, $x = 0$, the temperature is kept constant at $\theta_0°$, the rate of flow of heat at the origin becomes

$$W_0 = \frac{K\theta_0}{h\sqrt{\pi t}} \qquad (119)$$

Heaviside compared this problem with that given in the foregoing case of the infinite "ideal" cable, when

$$\frac{\partial e}{\partial x} = -iR \qquad (120)$$

and

$$\frac{\partial i}{\partial x} = -Cpe = -C\frac{de}{dt}$$

These two equations combined giving

$$\frac{\partial^2 e}{dx^2} = RC\frac{de}{dt} \text{ or } \frac{de}{dt} = \frac{1}{RC}\frac{d^2 e}{dx^2} \qquad (121)$$

Comparing equations (117) and (121), we note that $h^2 = 1/RC$ Comparing equations (118) and (120), we note that $K = 1/R$, $W = i$ and $\theta_0 = E\mathbf{1}$ and

$$\frac{K}{h} = \sqrt{\frac{C}{R}} = \left(\frac{C}{R}\right)^{\frac{1}{2}}$$

The operational solution of the current as given above was

$$i = \left(\frac{pC}{R}\right)^{\frac{1}{2}} \epsilon^{-nx} E\mathbf{1} = \left(\frac{C}{R}\right)^{\frac{1}{2}} Ep^{\frac{1}{2}} \epsilon^{-nx}\mathbf{1}$$

or, at the origin,

$$i_0 = \left(\frac{C}{R}\right)^{\frac{1}{2}} Ep^{\frac{1}{2}}\mathbf{1} \text{ for } x = 0$$

or substituting heat constants,

$$W_0 = \frac{K}{h}\theta_0 p^{\frac{1}{2}}\mathbf{1}$$

But in equation (119)

$$W_0 = \frac{K}{h} \cdot \frac{\theta_0}{\sqrt{\pi t}}$$

Heaviside concludes, therefore, that

$$p^{\frac{1}{2}}\mathbf{1} = \frac{1}{\sqrt{\pi t}} = (\pi t)^{-\frac{1}{2}}$$

Therefore, the current at the battery is

$$i_0 = \sqrt{\frac{C}{R\pi t}}E$$

The solution for the current when an alternator is connected to the cable is given in Chap. XXV.

We will now proceed to solve equation (116).

$$i = E\sqrt{\frac{C}{R}}p^{\frac{1}{2}}\left[1 - nx + \frac{n^2x^2}{\underline{|2}} - \cdots\right]\mathbf{1}$$

Substituting $n = \sqrt{pCR}$,

$$i = E\left[p^{\frac{1}{2}}\sqrt{\frac{C}{R}} - pCx + \frac{p^{\frac{3}{2}}C^{\frac{3}{2}}R^{\frac{1}{2}}x^2}{\underline{|2}} - \frac{p^2C^2Rx^3}{\underline{|3}} + \frac{p^{\frac{5}{2}}C^{\frac{5}{2}}R^{\frac{3}{2}}x^4}{4} + \cdots\right]\mathbf{1}$$

The terms involving even powers in p are zero, because $p\mathbf{1}$, $p^2\mathbf{1}$, etc., are zero; therefore,

$$i = E\sqrt{\frac{C}{\pi Rt}}\left[1 - \frac{CRx^2}{2\underline{|2}} + \frac{1\cdot 3C^2R^2x^4}{2\cdot 2\underline{|4}t^2} - \frac{1\cdot 3\cdot 5C^3R^3x^6}{2\cdot 2\cdot 2\underline{|6}t^4} + \cdots\right]$$

Let

$$y = \frac{CRx^2}{4t}$$

Then

$$i = E\sqrt{\frac{C}{\pi Rt}}\left[1 - y + \frac{y^2}{\underline{|2}} - \frac{y^3}{\underline{|3}} + \cdots\right] = E\sqrt{\frac{C}{\pi Rt}}\epsilon^{-\frac{CRx^2}{4t}}$$

This problem is of interest because the solution is a convergent series, although t appears in the denominators. The reason is that the development of ϵ^{-nx} introduces factorials.

CHAPTER XXV

PROBLEMS INVOLVING CONVERGENT AND ASYMPTOTIC SERIES

In his second volume on "Electromagnetic Theory," Heaviside gives, on page 42, the solution of a problem involving an infinitely long "ideal" cable connected to a battery through a coil. On page 40, he gives a similar problem where, instead of a coil, he inserts a condenser. In both cases, he obtains operational solutions involving fractional exponents of p. He proceeds to expand them in powers of $1/p$ and in powers of p. The expansions in powers of $1/p$ give convergent solutions and are conveniently used for small values of t. The asymptotic solution—if complete—is suitable for large values of t and is always subject to a certain error, which, however, can be determined.

In the second problem, the asymptotic expansion is the complete expansion; in the first problem, a term is added and no satisfactory reason can be found in his three volumes on "Electromagnetic Theory" or in his two volumes of *Electrical Papers*. To be sure, on page 159 of his first volume of *Papers* and on page 490 of his second volume of *Theory*, he discusses a similar problem, and these pages give a clue to his reason.

In connection with the study of operational calculus at Union College, and with the assistance of Dr. J. J. Smith and S. J. Haefner and E. W. Hamlin, we have found several methods which are applicable to problems involving only one variable t; in other words, to problems of infinite lines where the condition at the generator ($x = 0$) is desired. Undoubtedly, investigations along these lines have been made by others, but the writer is familiar only with the work of W. O. Pennell, which covers the same field as ours and gives identically the same results.

This field pertains to operators which are of the form

$$y = \frac{Y(p)}{Z(p)}E1$$

where $Z(p)$ is of a form which can be rationalized to contain only integral powers of p, and $Y(p)$ contains either integral or fractional powers of p or both.

The operational solutions for infinite lines where the condition at the generator is desired are of the form

$$y = \frac{Y(p)}{1 + (ap)^{\frac{n}{2}}} E\mathbf{1} \tag{122}$$

where n is any odd integer.

Procedure: Rationalize the denominator thus,

$$y = \frac{\left[1 - (ap)^{\frac{n}{2}}\right] Y(p)}{1 - (ap)^n} E\mathbf{1} \tag{123}$$

Convergent Solution.—In equation (122), divide through and obtain an infinite series with p^s in the denominator of each term. Some of the terms will contain even powers, others, fractional powers in p; and insert the values of the various $\frac{1}{p^s}\mathbf{1}$.

An alternative method, which is frequently preferable, is to break up equation (123) into two terms, one of which contains only even powers of p, the other, only fractional powers of p.

The first part is then solved by the expansion theorem; the second is expanded by division, as discussed above. The complete convergent solution is the sum of the two. Obviously, this solution does not contain the same terms as the first—but both give the same result. The second method usually tells the nature of the phenomenon, with far less work.

There is one thing to be careful about and that is to pick the right roots when $Z(p) = 0$. These roots should satisfy equation (122) as well as (123). To illustrate: If $n = 3$, the proper roots are

$$\frac{1}{a}\underline{|0^\circ}, \qquad \frac{1}{a}\underline{|+120^\circ}, \qquad \frac{1}{a}\underline{|-120^\circ}$$

not

$$\frac{1}{a}\underline{|0^\circ}, \qquad \frac{1}{a}\underline{|120^\circ}, \qquad \frac{1}{a}\underline{|240^\circ}$$

for $n = 5$, the roots are

$$\frac{1}{a}\underline{|0^\circ}, \qquad \frac{1}{a}\underline{|\pm 72^\circ}, \qquad \frac{1}{a}\underline{|\pm 144^\circ}$$

Asymptotic Solution.—Rule.—After equation (123) has been obtained, solve not only the part which contains even powers of p, but also the other part by the expansion theorem, omitting the $Y(0)$ term which is contained in the asymptotic series. The result obtained from these we have called the *additional term*, which often turns out to be zero. If it has real as well as imaginary terms, the imaginary part is rejected.[1]

The asymptotic series is obtained from equation (122) by division, arranging the division so that all p's are in the numerators of the different terms. Some terms will have even powers of p; these are all zero.

The methods given above seem to be the simplest of several which can be used.

The solution of a number of characteristic problems will now be given.

Problem.—An infinitely long "ideal" cable is connected to an alternator of voltage $e = E \sin \omega t$ at time $t = 0$. Find the equation for the current at the generator.

It was shown in Chap. XXIV that the operational solution when unit e.m.f. was supplied is

$$i = E\sqrt{\frac{C}{R}}p^{\frac{1}{2}}\mathbf{1}$$

The operational expression for $\sin \omega t 1$ is $\frac{p\omega}{p^2 + \omega^2}\mathbf{1}$. (See page 29)

Thus

$$i = E\sqrt{\frac{C}{R}}\,\omega\,\frac{p^{\frac{3}{2}}}{p^2 + \omega^2}\mathbf{1}$$

Convergent Solutions:

Since

$$\frac{1}{p^2 + \omega^2} = \frac{1}{p^2} - \frac{\omega^2}{p^4} + \frac{\omega^4}{p^6} - \cdots$$

$$i = E\omega\sqrt{\frac{C}{R}}[p^{-\frac{1}{2}} - \omega^2 p^{-\frac{5}{2}} + \omega^4 p^{-\frac{9}{2}} - \cdots]\mathbf{1}$$

$$= 2E\omega\sqrt{\frac{Ct}{R\pi}}\left[1 - \frac{(2\omega t)^2}{1\cdot3\cdot5} + \frac{(2\omega t)^4}{1\cdot3\cdot5\cdot7\cdot9} - \cdots\right]$$

---

[1] See article in *Jour. Franklin Inst.*, February, 1928.

as seen from equation (112a). Or we could obtain the solution by "shifting"

$$i = E\sqrt{\frac{C}{R}}\,\frac{1}{p^{\frac{1}{2}}}\,\frac{\epsilon^{j\omega t} - \epsilon^{-j\omega t}}{2j}\,1$$

Note that

$$p^{\frac{1}{2}}\epsilon^{j\omega t}1 = \epsilon^{j\omega t}(p + j\omega)^{\frac{1}{2}}\,1 = \epsilon^{j\omega t}[p^{\frac{1}{2}} + \tfrac{1}{2}p^{-\frac{1}{2}}j\omega - \cdots]1$$

and

$$p^{\frac{1}{2}}\epsilon^{-j\omega}1 = \epsilon^{-j\omega t}(p - j\omega)^{\frac{1}{2}}\,1 = \epsilon^{-j\omega t}[p^{\frac{1}{2}} - \tfrac{1}{2}p^{-\frac{1}{2}}j\omega + \cdots]1$$

The result becomes

$$i = E\sqrt{\frac{C}{R}}[A \sin \omega t + B \cos \omega t]$$

where

$$A = \frac{1}{\sqrt{\pi}}\left[t^{-\frac{1}{2}} + \frac{t^{\frac{3}{2}}\omega^2}{3\,\underline{|2}} - \frac{t^{\frac{7}{2}}\omega^4}{7\,\underline{|4}} + \cdots\right]$$

and

$$B = \frac{1}{\sqrt{\pi}}\left[\frac{\omega t}{\underline{|1}} - \frac{\omega^3 t^{\frac{5}{2}}}{5\,\underline{|3}} + \frac{\omega^5 t^{\frac{9}{2}}}{9\,\underline{|5}}\right]$$

Note that A and B converge rather rapidly. As t becomes larger and larger, A and B become more nearly equal and in the limit when $t = \infty$, $A = B = \sqrt{\frac{\omega}{2}}$.

Thus, the final current is

$$i_p = E\sqrt{\frac{C\omega}{2R}}(\sin \omega t + \cos \omega t)$$

It is of interest to note that there are several ways to get the convergent solution. We might have expressed

$$\sin \omega t = \omega t - \frac{\omega^3 t^3}{\underline{|3}} + \cdots$$

When these terms are operated on by $p^{\frac{1}{2}}$, we get

$$i = E\sqrt{\frac{C\omega}{\pi R}}\left[2\sqrt{\omega t} - \frac{2\cdot 2\cdot 2}{1\cdot 3\cdot 5}(\omega t)^{\frac{5}{2}} + \frac{2^5(\omega t)^{\frac{9}{2}}}{1\cdot 3\cdot 5\cdot 7\cdot 9} - \cdots\right]$$

Finally, by referring to the table of miscellaneous formulas and the comments on the use of Duhamel's integral, we get

$$i = \omega\sqrt{\frac{C}{\pi R}}\int_{u=0}^{u=t}\frac{\cos\omega u}{\sqrt{t-u}}du$$

We have given four different solutions, all having the same result and all suitable for small values of t. Incidentally, it is seen that $p^{\frac{1}{2}}\sin\omega t\mathbf{1}$ is a transient which eventually becomes $\sqrt{\omega}\sin(\omega t + 45^\circ)$, since $p = j\omega$ in the steady state.

Asymptotic Expansion:

$$\begin{aligned}
i &= E\sqrt{\frac{C}{R}}\frac{p^{\frac{1}{2}}\omega}{p^2+\omega^2}\mathbf{1}\\
&= E\sqrt{\frac{C}{R}}p^{\frac{1}{2}}\omega\left[\frac{1}{\omega^2} - \frac{p^2}{\omega^4} + \frac{p^4}{\omega^6} - \cdots\right]\mathbf{1}\\
&= E\sqrt{\frac{C}{R}}\omega\left[\frac{p^{\frac{1}{2}}}{\omega^2} - \frac{p^{\frac{5}{2}}}{\omega^4} + \frac{p^{\frac{9}{2}}}{\omega^6} - \cdots\right]\mathbf{1}\\
&= -E\sqrt{\frac{C}{R\pi t}}\left(\frac{1}{2\omega t} - \frac{1\cdot3\cdot5}{(2\omega t)^3} + \frac{1\cdot3\cdot5\cdot7\cdot9}{(2\omega t)^5}\cdots\right)\mathbf{1} \qquad (124)
\end{aligned}$$

Equation (124) is not the complete solution, because, in this case, there is an "additional term."

$$y = \frac{p^{\frac{1}{2}}}{p^2+\omega^2}$$

has a pair of imaginary roots, $p_1 = +j\omega = \omega|\underline{90^\circ}$ and $p_2 = -j\omega = \omega\,\lfloor\underline{-90^\circ}$, which are obtained in the usual way by making $Z_{(p)} = 0$

$$p\frac{dZ}{dp} = 2p^2$$

$$\frac{Y_{(p)}}{p\frac{dZ}{dp}} = \frac{p^{\frac{1}{2}}}{2p^2} = \frac{1}{2p^{\frac{3}{2}}}$$

$$\begin{aligned}
\therefore \left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p=p_1}, &= \frac{1}{2\sqrt{\omega}}\lfloor\underline{-45^\circ} = \frac{1}{2\sqrt{\omega}}(\cos 45^\circ - j\sin 45^\circ)\\
&= \frac{1}{2\sqrt{2\omega}}(1-j)
\end{aligned}$$

Similarly,

$$\left[\frac{Y_{(p)}}{p\frac{dZ}{dp}}\right]_{p=p_2} = \frac{1}{2\sqrt{2\omega}}(1+j)$$

$$\therefore y = \frac{1}{2\sqrt{2\omega}}[(1-j)\epsilon^{j\omega t} + (1+j)\epsilon^{-j\omega t}]$$

$$= \frac{1}{\sqrt{2\omega}}(\cos\omega t + \sin\omega t)$$

and the complete asymptotic solution is

$$i = E\sqrt{\frac{C\omega}{2R}}(\cos\omega t + \sin\omega t) - E\sqrt{\frac{C}{R\pi t}}\left(\frac{1}{2\omega t} - \frac{1\cdot3\cdot5}{(2\omega t)^2} + \cdots\right)$$

Problem.—A condenser C_1 is connected between the battery and an infinitely long "ideal" cable, as shown in Fig. 50. Find the potential difference at the right-hand side of the condenser.

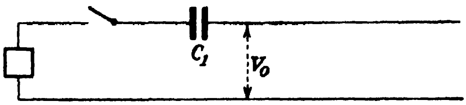

FIG. 50.

It has been shown that the impedance operator which replaces the cable is $(R/pC)^{\frac{1}{2}}$. Thus, the total impedance at the battery is

$$\left(\frac{R}{pC}\right)^{\frac{1}{2}} + \frac{1}{pC_1}$$

Therefore, the current at the battery is

$$i = \frac{E}{\left(\frac{R}{pC}\right)^{\frac{1}{2}} + \frac{1}{pC_1}}\mathbf{1}$$

The voltage consumed by the condenser is i/pC_1. Thus,

$$V_0 = E\mathbf{1} - \frac{E}{\left(\frac{R}{pC}\right)^{\frac{1}{2}} + \frac{1}{pC_1}}\,\frac{1}{pC_1}\mathbf{1}$$

This can be brought into the following form:

$$V_0 = E\frac{h^{\frac{1}{2}}p^{\frac{1}{2}}}{h^{\frac{1}{2}}p^{\frac{1}{2}} + 1}\mathbf{1}, \text{ where } h = \frac{RC_1^2}{C}$$

Convergent Solution:

$$V_0 = Eh^{\frac{1}{2}}p^{\frac{1}{2}}\left[\frac{1}{h^{\frac{1}{2}}p^{\frac{1}{2}}} - \frac{1}{hp} + \frac{1}{h^{\frac{3}{2}}p^{\frac{3}{2}}} - \cdots\right]1$$

$$= E\left[1 - \frac{1}{h^{\frac{1}{2}}p^{\frac{1}{2}}} + \frac{1}{hp} - \frac{1}{h^{\frac{3}{2}}p^{\frac{3}{2}}} + \cdots\right]1$$

Here we have one series of integral powers in $1/p$. This part sums up to $E\epsilon^{\frac{t}{h}}$. In addition, we have a series in fractional powers of $1/p$.

The latter becomes

$$-2E\left(\frac{t}{h\pi}\right)^{\frac{1}{2}}\left[1 + \frac{2t}{1\cdot3h} + \frac{1}{1\cdot3\cdot5}\left(\frac{2t}{h}\right)^2 + \cdots\right.$$

Thus, the convergent solution is

$$V_0 = E\left\{\epsilon^{\frac{t}{h}} - 2\left(\frac{t}{h\pi}\right)^{\frac{1}{2}}\left[1 + \frac{2t}{h}\cdot\frac{1}{1\cdot3} + \left(\frac{2t}{h}\right)^2\frac{1}{1\cdot3\cdot5} + \cdots\right]\right\}$$

Asymptotic Expansion:

$$V_0 = Eh^{\frac{1}{2}}p^{\frac{1}{2}}[1 - h^{\frac{1}{2}}p^{\frac{1}{2}} + hp - h^{\frac{3}{2}}p^{\frac{3}{2}} + h^2p^2 - \cdots]1$$
$$= E[h^{\frac{1}{2}}p^{\frac{1}{2}} - hp + h^{\frac{3}{2}}p^{\frac{3}{2}} - h^2p^2 + \cdots]1$$

The even powers in p contribute nothing, thus:

$$V_0 = E[h^{\frac{1}{2}}p^{\frac{1}{2}} + h^{\frac{3}{2}}p^{\frac{3}{2}} + \cdots]1$$
$$= E\left(\frac{h}{t\pi}\right)^{\frac{1}{2}}\left[1 - \frac{1\cdot h}{2t} + 1\cdot3\left(\frac{h}{2t}\right)^2 - 1\cdot3\cdot5\left(\frac{h}{2t}\right)^3 + \cdots\right]$$

The question is whether or not this is the complete asymptotic solution.

In this case, it is the complete solution, because when the denominator is rationalized we get

$$V_0 = E\frac{h^{\frac{1}{2}}p^{\frac{1}{2}}}{h^{\frac{1}{2}}p^{\frac{1}{2}} + 1} = E\frac{h^{\frac{1}{2}}p^{\frac{1}{2}}(h^{\frac{1}{2}}p^{\frac{1}{2}} - 1)}{hp - 1}$$

$Z_{(p)} = 0$ gives $p = 1/h$, which is real. When this is used, the numerator becomes zero, thus: $V'_0 = 0$.

Problem.—Consider, next, a similar problem of an infinitely long "ideal" cable, where the condenser is replaced by a coil of inductance L_1, as in Fig. 51. It is readily found that

$$V_0 = \frac{E}{1 + g^{\frac{3}{2}}p^{\frac{3}{2}}}\mathbf{1} \quad \text{where } g^{\frac{3}{2}} = \frac{C^{\frac{1}{2}}L_1}{R^{\frac{1}{2}}}$$

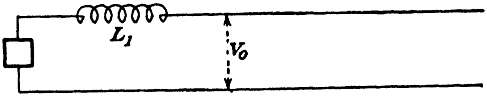

FIG. 51.

Convergent Solution:

$$V_0 = E\left[\frac{1}{g^{\frac{3}{2}}p^{\frac{3}{2}}} - \frac{1}{g^3p^3} + \frac{1}{g^{\frac{9}{2}}p^{\frac{9}{2}}} - \cdots\right]\mathbf{1}$$
$$= E\frac{4\pi}{3}\left(\frac{t}{g\pi}\right)^{\frac{3}{2}}\left[1 + \left(\frac{2t}{g}\right)^3\frac{1}{5\cdot7\cdot9} + \left(\frac{2t}{g}\right)^6\frac{1}{5\cdot7\cdot9\cdot11\cdot13\cdot15}\cdots\right]$$
$$- E\left[\left(\frac{t}{g}\right)^3\frac{1}{3} + \left(\frac{t}{g}\right)^6\frac{1}{\underline{|6}} + \cdots\right]$$

This solution is complete but does not show directly that there is an oscillation. This can be brought out by rewriting the operational equation with rationalized denominator. It is

$$V_0 = \frac{E}{1 + g^{\frac{3}{2}}p^{\frac{3}{2}}}\mathbf{1} = \frac{E(1 - g^{\frac{3}{2}}p^{\frac{3}{2}})}{1 - g^3p^3}\mathbf{1} = E(A + B)$$

where

$$A = \frac{1}{1 - g^3p^3} \text{ and } B = -\frac{g^{\frac{3}{2}}p^{\frac{3}{2}}}{1 - g^3p^3}$$

A is solved by the expansion theorem. $Z(p) = 0$ gives three roots:

$$\frac{1}{g} \text{ and } \frac{1}{g}\angle\pm120°$$

The result is

$$A = \left[1 - \frac{1}{3}\epsilon^{\frac{t}{g}} - \frac{2}{3}\epsilon^{-\frac{t}{2g}}\cos\frac{\sqrt{3}}{2g}t\right]$$

B when expanded in powers of $1/p$ gives the same result as we obtained in the expansion given above. It is

$$B = \frac{4\pi}{3}\left(\frac{t}{g\pi}\right)^{\frac{3}{2}}\left[\mathbf{1} + \left(\frac{2t}{g}\right)^3\frac{1}{5\cdot7\cdot9} + \cdots\right]$$

It is interesting to note that

$$\left(\frac{t}{g}\right)^3\frac{1}{\underline{|3}} + \left(\frac{t}{g}\right)^6\frac{1}{\underline{|6}} + \cdots = -\left(1 - \frac{1}{3}\epsilon^{\frac{t}{g}} - \frac{2}{3}\epsilon^{-\frac{t}{2g}}\cos\frac{\sqrt{3}}{2g}t\right)$$

Asymptotic Expansion:

$$V_0 = E[1 - g^{\frac{1}{2}}p^{\frac{1}{2}} + g^3p^3 - g^{\frac{7}{2}}p^{\frac{7}{2}} + \cdots]1$$

The even powers in p contribute nothing. Thus,

$$V_0 = E\left\{1 + \frac{\pi}{2}\left(\frac{g}{\pi t}\right)^{\frac{1}{2}}\left[1 - 1\cdot3\cdot5\cdot7\left(\frac{g}{2t}\right)^3 + 1\cdot3\cdot5\cdot7\cdot9\;11\cdot13\left(\frac{g}{2t}\right)^6 + \cdots\right]\right\}$$

To find the "additional term,"

$$V_0' = E\frac{(1 - g^{\frac{1}{2}}p^{\frac{1}{2}})}{1 - g^3p^3}$$

$p\frac{dZ}{dp} = -3g^3p^3$, which for $p^3 = \frac{1}{g^3}$ gives -3.

$$1 - g^{\frac{1}{2}}p^{\frac{1}{2}}\left(\text{for } p = \frac{1}{g}\right) = 0$$

$$1 - g^{\frac{1}{2}}p^{\frac{1}{2}}\left(\text{for } p = \frac{1}{g}\underline{|+120^\circ}\right) = 1 - 1\underline{|180^\circ}$$

$$1 - g^{\frac{1}{2}}p^{\frac{1}{2}}\left(\text{for } p = \frac{1}{g}\underline{|-120^\circ}\right) = 1 - 1|-180^\circ$$

$$\therefore V_0' = \frac{1+1}{-3}\epsilon^{(-\alpha+j\beta)t} + \frac{1+1}{-3}\epsilon^{(-\alpha-j\beta)t}$$

$$= -\frac{4}{3}\epsilon^{-\alpha t}\cos\beta t. \quad \text{where } \alpha = \frac{1}{2g} \text{ and } \beta = \frac{\sqrt{3}}{2g}$$

The complete asymptotic solution is, then, the asymptotic expansion $V_0 + V_0'$.

Problem.—An infinitely long line, having constants R, L, and C, is connected to a storage battery of voltage E. Find the current at the battery.

The operational solution for any infinitely long line has been shown to be

$$i = \frac{Y}{n}E1 = \frac{pC}{\sqrt{pC(R + pL)}}E1 = \left(\frac{pC}{R + pL}\right)^{\frac{1}{2}}E1$$

Let

$$\frac{R}{2L} = \alpha \text{ then } i = \frac{C^{\frac{1}{2}}}{L^{\frac{1}{2}}}\frac{p^{\frac{1}{2}}}{(p + 2\alpha)^{\frac{1}{2}}}E1$$

Convergent Solution:

$$i = \frac{C^{\frac{1}{2}}}{L^{\frac{1}{2}}} \frac{E}{\left(1 + \frac{2\alpha}{p}\right)^{\frac{1}{2}}} \mathbf{1} = E\frac{C^{\frac{1}{2}}}{L^{\frac{1}{2}}}\left[\mathbf{1} - \frac{2\alpha}{2p} + \frac{1\cdot 3}{2\cdot 4}\left(\frac{2\alpha}{p}\right)^2 + \cdots\right]\mathbf{1}$$

$$= E\frac{C^{\frac{1}{2}}}{L^{\frac{1}{2}}}\left[1 - \frac{\alpha}{p} + \frac{1\cdot 3}{\underline{|2}}\left(\frac{\alpha}{p}\right)^2 - \frac{1\cdot 3\cdot 5}{\underline{|3}}\left(\frac{\alpha}{p}\right)^3 + \cdots\right]\mathbf{1}$$

$$= E\frac{C^{\frac{1}{2}}}{L^{\frac{1}{2}}}\left[1 - \alpha t + \frac{1\cdot 3}{(\underline{|2})^2}(\alpha t)^2 - \frac{1\cdot 3\cdot 5}{(\underline{|3})^2}(\alpha t)^3 + \cdots\right]\mathbf{1}$$

This is a convergent series suitable for small values of αt.

Asymptotic Expansion:

$$E\frac{C^{\frac{1}{2}}}{L^{\frac{1}{2}}(2\alpha)^{\frac{1}{2}}}\left[p^{\frac{1}{2}} - \frac{1}{2}\frac{p^{\frac{3}{2}}}{2\alpha} + \frac{1\cdot 3}{2\cdot 4}\frac{p^{\frac{5}{2}}}{(2\alpha)^2} - \cdots\right]\mathbf{1}$$

$$= E\frac{C^{\frac{1}{2}}}{L^{\frac{1}{2}}}\frac{1}{\sqrt{2\alpha\pi t}}\left[1 + \frac{1^2}{8\alpha t} + \frac{1^2\cdot 3^2}{\underline{|2}(8\alpha t)^2} + \frac{1^2\cdot 3^2\cdot 5^2}{\underline{|3}(8\alpha t)^3} + \cdots\right]$$

This is the complete solution. The "additional term" is zero, as can easily be seen.

Characteristics of Asymptotic Series Illustrated Numerically in This Series.—Assign certain values to the independent variable t, or perhaps, more conveniently to $8\alpha t$ in the above series. Let $8\alpha t$ be equal to 8.

Then the series becomes

$$1 + \frac{1}{8} + \frac{9}{2\cdot 64} + \frac{225}{6\cdot 512} + \cdots$$

$$= 1 + 0.125 + 0.0351 + 0.0735 + \cdots$$

Note that the successive terms are smaller up to the third term. After that they increase.

Heaviside says: "Seek the place where the smallest term occurs, then the error where the series is summed up to this term is less than the smallest term."

In this case, the sum is 1.1601 with an error less than 0.0351.

The larger the value of $8\alpha t$ the smaller is the percentage error.

CHAPTER XXVI

DIFFICULTIES OF INTERPRETING POWER-SERIES DEVELOPMENTS

It is not always possible to see the physical meaning of a power series. It may give a far more complete answer to a problem than is suspected at a superficial glance.

As an illustration, consider an alternating e.m.f. $E\sin \omega t$ impressed on an inductive circuit.

The operational solution is, as shown in Chap. VI,

$$i = \frac{p\omega}{(p^2 + \omega^2)(r + pL)} E\mathbf{1}$$

If this is expanded by "algebrizing," we get

$$\begin{aligned} i &= \frac{\omega E}{L}\left[\frac{1}{p^2} - \frac{\alpha}{p^3} + \frac{\alpha^2 - \omega^2}{p^4} \right. \\ &\qquad \left. - \frac{\alpha(\alpha^2 - \omega^2)}{p^5} + \frac{\omega^4 - \alpha^2\omega^2 + \alpha^2}{p^6} \cdots \right]\mathbf{1} \\ &= \frac{E\omega}{L}\left[\frac{t^2}{\underline{|2}} - \frac{t^3\alpha}{\underline{|3}} + \frac{t^4}{\underline{|4}}(\alpha^2 - \omega^2) \right. \\ &\qquad \left. - \frac{t^5}{\underline{|5}}\alpha(\alpha^2 - \omega^2) + \frac{t^6}{\underline{|6}}(\omega^4 - \alpha^2\omega^2 + \alpha^2) \cdots \right] \end{aligned} \qquad (125)$$

It is not readily seen or generally known that this series gives

$$i = \frac{E}{Z}[\sin(\omega t - \beta) + \epsilon^{-\alpha t}\sin\beta] \qquad (126)$$

which is the proper result not only for the permanent term but also for the transient.

In this case, $\alpha = \frac{r}{L}$ and $\tan\beta = \frac{\omega L}{r} = \frac{\omega}{\alpha}$. To prove it, expand equation (126) in a power series.

CHAPTER XXVII

LINEAR FLOW OF HEAT

The differential equations pertaining to heat problems in one dimension are identical with those of the "ideal" cable, if the heat losses due to convection and radiation are neglected.

Let Fig. 52 represent a section of a conducting wall, one part of which is, metal, the other part, cork; and let us assume that a definite amount of heat is flowing to each unit area per unit time. Find the temperature at any point.

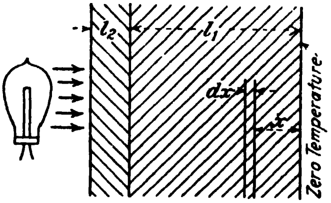

FIG. 52.

Let V be the temperature above zero at any point in the material.

Let i be the amount of heat given to each unit area per unit time.

Let R be the heat resistance of the material; it is the temperature drop per unit length when unit heat flows through the material and steady condition has been reached. Thus, Ri is the temperature drop per unit length.

Let C be the heat capacity per unit volume; it is the number of units of heat which are required to raise the temperature 1°. The temperature increases as x increases. The rate of increase is $\partial V/\partial x$; thus, the difference in temperature over a distance dx is $\frac{\partial V}{\partial x}\,dx$, and this difference, as has been stated, is equal to $iRdx$.

$$\therefore i = \frac{1}{R}\frac{\partial V}{\partial x} \qquad (127)$$

There is a difference in flow of heat on each side of the element dx due to the heat absorbed.

This difference is $\frac{\partial i}{\partial x}dx$. But heat capacity is, by definition,

$$C = \frac{\text{total heat absorbed}}{\text{temperature rise}}$$

$$= \frac{\text{heat absorbed per unit time} \times \text{time}}{\text{temperature difference}}$$

Thus,

$$C dx = \frac{\frac{\partial i}{\partial x} dx dt}{dV} \text{ or } \frac{\partial i}{\partial x} = C \frac{dV}{dt} \tag{128}$$

From equation (127),

$$\frac{\partial i}{\partial x} = \frac{1}{R} \frac{\partial^2 V}{\partial x^2}$$

$$\therefore \frac{\partial^2 V}{\partial x^2} = CR \frac{dV}{dt} \tag{129}$$

$$\therefore \frac{d^2 V}{dx^2} = pCRV = n^2 V \tag{130}$$

From Chap. XV,

$$V = K_1 \cosh nx + K_2 \sinh nx$$

$$i = \frac{Y}{n}[K_1 \sinh nx + K_2 \cosh nx]$$

where $Y = pC$ and $n^2 = pCR$.

If for $x = 0$ (outside wall), $V = 0$, then K_1 must be zero.

$$\therefore V = K_2 \sinh nx$$

and

$$i = \frac{Y}{n} K_2 \cosh nx$$

From an electrical point of view, we have really two ideal cables of different constants n_1 and n_2. These cables are connected in series and the far end is short circuited (since for $x = 0$, $V = 0$). At the generator, a fixed current $I1$ is supplied and the problem is to find the voltage at any point in the system.

Referring to Chap. XIX, we substitute for the right-hand cable an impedance z'_2 placed at the end of the first cable. This impedance is

$$z'_2 = \frac{V}{i} = \frac{n_1}{Y_1}\frac{\sinh n_1l_1}{\cosh n_1l_1} \tag{131}$$

We have, then, a single cable of length l_2 with an impedance z'_2 at the end of the line. The general equations for this line are

$$V = K_1 \cosh n_2x + K_2 \sinh n_2x$$

$$i = \frac{Y_2}{n_2}[K_1 \sinh n_2x + K_2 \cosh n_2x]$$

In this case, x is measured from the right-hand edge of the heavier shaded material toward the left. For

$$x = 0,\ V = i_2z'_2;\ \therefore i_2z'_2 = (K_1 + 0) = K_1$$

therefore,

$$V = i_2z'_2 \cosh n_2x + K_2 \sinh n_2x$$

$$i = \frac{Y_2}{n_2}[i_2z'_2 \sinh n_2x + K_2 \cosh n_2x]$$

For

$$x = 0, \qquad i = i_2$$

Thus,

$$i_2 = \frac{Y_2}{n_2}K_2 \text{ or } K_2 = \frac{n_2i_2}{Y_2}$$

For

$$x = l_2, \qquad i = I1$$

$$\therefore I1 = \frac{Y_2}{n_2}\left[i_2z'_2 \sinh n_2l_2 + \frac{n_2i_2}{Y_2} \cosh n_2l_2\right]$$

$$\therefore i_2 = \frac{In_2}{Y_2\left[z'_2 \sinh n_2l_2 + \frac{n_2}{Y_2} \cosh n_2l_2\right]}1$$

Thus, the voltage at the junction between the two materials is

$$i_2z'_2 = \frac{Iz'_2n_2}{Y_2\left[z'_2 \sinh n_2l_2 + \frac{n_2}{Y_2} \cosh n_2l_2\right]}1$$

and the voltage at any distance x from the end of the short-circuited cable is, from Chap. XV (equation (57)).

$$V = \frac{In_2z_2}{Y_2\left[z'_2 \sinh n_2l_2 + \frac{n_2}{Y_2} \cosh n_2l_2\right]} \times \frac{\sinh n_1x}{\sinh n_1l_1}1$$

After substituting the values for z'_2, z_1, and z_2, we have

$$V = \frac{IR_1R_2 \sin m_1x}{R_2m_1 \cos m_1l_1 \cos m_2l_2 - R_1m_2 \sin m_1l_1 \sin m_2l_2}\mathbf{1}$$

$$Z_{(p)} = 0 \text{ gives } \tan m_1l_1 \tan m_2l_2 = \sqrt{\frac{C_1R_2}{C_2R_1}}$$

The solution is very cumbersome and will not be attempted.

To introduce fractional differentiation, another and simpler problem will be considered. Assume that the wall in Fig. 52 has no metal lining, so that we may deal with the conduction of heat by one substance only.

From the general equations for a short-circuited "ideal" cable of finite length, we have

$$e = e_0\frac{\sinh nx}{\sinh nl}$$

and

$$i = e_0\frac{Y}{n}\frac{\cosh nx}{\sinh nl}$$

Again let the flow of heat applied at $x = l$ be $I\mathbf{1}$. Then

$$e_0 = \frac{n}{Y}\frac{\sinh nl}{\cosh nl}I\mathbf{1} \text{ or, since } \frac{Z}{n} = \frac{n}{Y}$$

$$e = \frac{R}{m}\frac{\sin ml \sin mx}{\cos ml \sin ml}I\mathbf{1} = \frac{R}{m}\frac{\sin mx}{\cos ml}I\mathbf{1} \tag{132}$$

The first question is can the expansion theorem be used. Is $p = 0$ a root? If it were, then as p approaches zero, equation (132) should approach infinity. That is not the case. As p approaches zero, equation (132) becomes IRx. Therefore, the expansion theorem can be used.

$$Z_{(p)} = 0 \text{ gives } \cos ml = 0 \text{ or } ml = \frac{2s-1}{2}\pi$$

$p\frac{dZ}{dp}$ becomes, after the roots of p have been substituted and after some transformations, $\frac{m^2l}{2}(-1)^s$. Since $\cos ml = 0$ and $\sin ml = (-1)^s$

$$\frac{Y_{(0)}}{Z_{(0)}} = IRx$$

$$\therefore e = IR\left[x + \sum_{s=1}^{s=\infty}(-1)^s\frac{8}{\pi^2}l\frac{\sin mx}{(2s-1)^2}\epsilon^{-\frac{m^2}{CR}t}\right] \tag{133}$$

At the source of heat, $x = l$ and

$$e = IRl\left[1 + \frac{8}{\pi^2}\sum_{s=1}^{s=\infty}\frac{\epsilon^{-\frac{m^2t}{CR}}}{(2s-1)^2}\right]$$

where

$$m = \frac{2s-1}{2}\frac{\pi}{l}$$

The same problem will now be considered from the point of view of an infinitely thick wall or an infinitely long "ideal" cable.

Referring to Chap. XXIV and measuring y from the heated side,

$$e = A\epsilon^{-ny} \qquad\qquad i = \frac{Y}{n}A\epsilon^{-ny}$$

For

$$y = 0 \qquad i = I\mathbf{1} \qquad \therefore A = \frac{n}{Y}I\mathbf{1}$$

and

$$e = \frac{n}{Y}\epsilon^{-ny} \qquad\qquad I\mathbf{1} = \frac{R}{n}\epsilon^{-ny}\,I\mathbf{1}$$

$$= \frac{RI}{n}\left(1 - ny + \frac{n^2y^2}{\underline{|2}} - \cdots\right)\mathbf{1}$$

Since $n = \sqrt{pCR}$,

$$e = RI\left[\frac{p^{-\frac{1}{2}}}{\sqrt{CR}} - y + \frac{\sqrt{CR}y^2}{\underline{|2}}p^{\frac{1}{2}} - \frac{CRy^3}{\underline{|3}}p + \cdots\right]\mathbf{1}$$

or

$$e = I\sqrt{\frac{R}{\pi C}}\left[2t^{\frac{1}{2}} - y\sqrt{\pi CR} + \frac{CRy^2}{\underline{|2}}t^{-\frac{1}{2}} - \frac{C^2R^2y^4}{2\underline{|4}}t^{-\frac{3}{2}}\cdots\right]$$

At the heated surface ($y = 0$),

$$e_0 = I\sqrt{\frac{4Rt}{\pi C}}$$

Using the c.g.s. system and cork,

$$R = \backsim 10{,}000 \text{ and } C = \backsim 0.07$$

1 watt per square inch corresponds to 0.256/1,000 gram calorie per second per cm.^{2}

CHAPTER XXVIII

THE NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS[1]

A remarkable article by Germinal Dandelin was published in 1826 in the *Memoirs* of the Royal Academy of Sciences of Brussels. It describes a masterly device for obtaining the roots of an equation. Unfortunately, Dandelin's research had the misfortune to be buried in the ponderous tomes of a royal academy.

Later, the Academy of Science of Berlin offered a prize for the discovery of a practical method of computing imaginary roots. The prize was awarded to Carl Heinrich Graeffe in 1837. His method was known as the *Graeffe method,* and he employed the same principle as Dandelin. It is of great use, especially in the case of equations possessing complex roots. Preliminary determination of their approximate position is unnecessary, and for algebraic equations all roots are found at once.

Its principle is to form a new equation whose roots are some high power—say, the two hundred fifty-sixth of the roots of the given equation. The law by which the new equations are constructed from the original is exceedingly simple. If x_1, x_2, x_3 . . . are the roots of the given equation, then x_1^{256}, x_2^{256}, x_3^{256} $\cdots$ will be the roots of the constructed equation. The latter roots are widely separated, for if x_1, were twice x_2, then $x_1,^{256}$ would be 2^{256} times x_2^{256}, and an equation whose roots are widely separated can be solved at once numerically.

Suppose that it is required to solve the equation:

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_n = 0 \qquad (134)$$

Assume, for the present, that the roots are real and unequal and that they are $-a$, $-b$, $-c$, $-d$ $\cdots$, the order being that of descending numerical magnitude, so that $|a| > |b| > |c| > |d| \cdots$

---

[1] The application to electrical problems as well as the mathematical theory for the solution of algebraic and transcendental equations is here given. The basic theory is taken from Whittaker and Robinson, "Calculus of Observations."

The values $a, b, c, d \ldots$, which are the roots of the equation reversed in sign, will be called the *Encke roots.*

Suppose $[a] = a + b + c + \cdots$ *i.e.*, the sum of the Encke roots and $[ab] = ab + ac + bc + \cdots$ *i.e.*, the sum of the products of the Encke roots taken two at a time, and so on. Also,

$$[a^m] = a^m + b^m + c^m + \cdots$$
$$[a^m b^m] = a^m b^m + a^m c^m + b^m c^m + \cdots$$

Now we can write equation (134) as

$$x^n + [a]x^{n-1} + [ab]x^{n-2} + [abc]x^{n-3} + \cdots [abcd \cdots] = 0 \quad (135)$$

and the equation whose roots are the m^{th} power of the roots of the given equation will be, m being even,

$$x^n + [a^m]x^{n-1} + [a^m b^m]x^{n-2} + [a^m b^m c^m]x^{n-3} + \cdots = 0 \quad (136)$$

It is necessary to construct equation (136) when equation (135) is given and m is some prescribed number.

In practice, m is a large number in equation (136) which is ultimately formed, but we do not attempt to construct this equation at a single step; instead, we first take $m = 2$, *i.e.*, we form a new equation whose Encke roots are the squares of the Encke roots of the original equation; then, having done this, we repeat the process, forming a new equation whose Encke roots are the squares of the equation last obtained—that is, the fourth powers of the Encke roots of the given equation—and so on.

Thus, our immediate problem is to construct equation (136) when equation (135) is given and m has the value 2. This is done as follows: Rearrange (134) so that the terms containing the even powers of x are on one side and the terms containing the odd powers are on the other; assume n even:

$$(x^n + a_2x^{n-2} + a_4x^{n-4} + \cdots) = (-a_1x^{n-1} - a_3x^{n-3} - a_5x^{n-5} - \cdots)$$

squaring both sides,

$$(x^n + a_2x^{n-2} + a_4x^{n-4} + \cdots)^2 = (a_1x^{n-1} + a_3x^{n-3} + a_5x^{n-5} + \cdots)^2$$

expanding,

$$x^{2n} + a_2x^{2n-2} + a_4x^{2n-4} + a_2x^{2n-2} + a_2{}^2x^{2n-4} + a_2a_4x^{2n-6} + \\ a_4x^{2n-4} + a_2a_4x^{2n-6} + a_4{}^2x^{2n-8} + \cdots \\ = a_1{}^2x^{2n-2} + a_1a_3x^{2n-4} + a_1a_5x^{2n-6} + a_1a_3x^{2n-4} + a_3{}^2x^{2n-6} \\ + a_3a_5x^{2n-8} + \cdots$$

Now let $-y = x^2$

$$(-y)^n + a_2(-y)^{n-1} + a_4(-y)^{n-2} + a_2(-y)^{n-1} + a_2{}^2(-y)^{n-2} + \\ a_2a_4(-y)^{n-3} + a_4(-y)^{n-2} + a_2a_4(-y)^{n-3} \quad \cdots \\ = a_1{}^2(-y)^{n-1} + a_1a_3(-y)^{n-2} + a_1a_5(-y)^{n-3} + a_1a_3(-y)^{n-2} \\ +a_3{}^2(-y)^{n-3} + a_5a_3(-y)^{n-4} + \cdots$$

and, since n is even, this becomes

$$y^n - a_2y^{n-1} + a_4y^{n-2} - a_2y^{n-1} + a_2{}^2y^{n-2} - a_2a_4y^{n-3} + a_4y^{n-2} \\ -a_2a_4y^{n-3} + \cdots \\ = -a_1{}^2y^{n-1} + a_1a_3y^{n-2} - a_1a_5y^{n-3} + a_1a_3y^{n-2} - a_3{}^2y^{n-3} + \\ a_5a_3y^{n-4} + \cdots$$

collecting, we have

$$y^n + (a_1{}^2 - 2a_2)y^{n-1} + (a_2{}^2 - 2a_1a_3 + 2a_4)y^{n-2} + \cdots = 0 \qquad (137)$$

but the roots of (134) are $-a,\ -b,\ -c \cdots$
$\therefore$ the roots of (137) are $-a^2,\ -b^2,\ -c^2 \cdots$

Thus, writing x in place of y, the equation whose Encke roots are the squares of the Encke roots of (134), is

$$x^n + \left.\begin{array}{r} a_1{}^2 \\ -2a_2 \end{array}\right\} x^{n-1} + \left.\begin{array}{r} a_2{}^2 \\ -2a_1a_3 \\ +2a_4 \end{array}\right\} x^{n-2} + \left.\begin{array}{r} a_3{}^2 \\ -2a_2a_4 \\ +2a_1a_5 \\ -2a_6 \end{array}\right\} x^{n-3} + \left.\begin{array}{r} a_4{}^2 \\ -2a_3a_5 \\ +2a_2a_6 \\ -2a_1a_7 \\ +2a_8 \end{array}\right\} x^{n-4} + \cdots = 0 \qquad (138)$$

Stating equation (138) in words, we arrive at the law of formation. The coefficient of any power of x is formed by adding to the square of the corresponding coefficient in the original equation the doubled product of every pair of coefficients which stand equally far from it on either side, these products being taken with alternately negative and positive signs.

Now repeat the process to equation (138) and obtain an equation whose Encke roots are the fourth power of equation (134).

The next stage yields an equation whose Encke roots are the eighth power of (134), and so on.

Suppose $m = 64$ or 128 or 256; since $a > b$, then a^m is enormously larger than b^m or c^m or d^m, and thus the sum $[a^m]$ bears to its first term a^m a ratio which is very near to unity. Similarly, $[a^m b^m] \cong a^m b^m$, or, more precisely, since $a > b > c > d$

$$[a^m]/a^m = (1 + \epsilon) \text{ where } [a^m] = a^m + b^m + c^m \cdots$$

and ϵ is very small.

$$\begin{aligned} i.e., \log [a^m] &= m \log |a| + \log (1 + \epsilon) \\ \therefore \log |a| &= \frac{1}{m} \log [a^m] - \frac{1}{m} \log (1 + \epsilon) \end{aligned}$$

We have calculated $[a^m]$ and can neglect $1/m \log (1 + \epsilon)$. NOTE.—m is made large enough so that this is possible. Thus, $|a|$ is determined. Next

$$\begin{aligned} [a^m b^m] &= a^m b^m (1 + \gamma) \\ \log a^m b^m + \log (1 + \gamma) &= \log [a^m b^m] \\ \log ab &= \frac{1}{m} \log [a^m b^m] - \frac{1}{m} \log (1 + \gamma) \\ \log |b| &= \frac{1}{m} \log [a^m b^m] - \frac{1}{m} \log [a^m] \end{aligned}$$

thus, $|b|$ is determined, and so on. In order to find the correct sign, we may substitute the root in the original equation.

An all-important question is when to stop in our doubling of m. The time to stop is when another doubling of m would give a result not different in the digits we wish to include from the roots which would be obtained by stopping at once—that is to say, when the coefficients $[a^{2m}]$, $[a^{2m} b^{2m}]$ of the new equation are practically nothing but the squares of the corresponding coefficients $[a^m]$, $[a^m b^m]$ in the equation already obtained. This point will show up clearly, however, in a numerical illustration.

Suppose we take a simple quadratic equation for illustration: $(x + 2)(x + 1) = 0$. Here the roots are -2, -1, and the Encke roots are 2, 1.

The following tabulation is convenient:

$$x^2 + 3x + 2 = 0$$

	x^2	x	c
p	1	3	2
Square of coefficient	1	9	4
doubled products		-4	
p^2	1	5	4
	1	25	16
		-8	
p^4	1	17	16
	1	289	256
		-32	
p^8	1	257	256
	1	6.6×10^4	6.55×10^4
		-0.05×10^4	
p^{16}	1	6.55×10^4	6.55×10^4

p denotes the original equation.

p^2 is the equation whose Encke roots are the squares of the Encke roots of p.

p^4 is the equation whose Encke roots are the squares of the Encke roots of p^2.

p^{16}	1	6.55×10^4	6.55×10^4
	1	42.8×10^8	42.8×10^8
		-0.00131×10^8	

p^{32} has coefficients which are practically the squares of the coefficients of p^{16} to the number of figures we wish to carry:

p^{32}	1	42.80×10^8	42.8×10^8
		or $(6.55 \times 10^4)^2$	

Thus, we will stop at $m = 16$.

$x^2 + 6.55 \times 10^4 x + 6.55 \times 10^4 = 0$ is the equation whose Encke roots are the sixteenth power of the Encke roots of the original equation.

$$a^{16} = [a^{16}] = 6.55 \times 10^4$$

$$\log |a| = \frac{1}{16} \log 6.55 \times 10^4 = \frac{1}{16}[\log\ 10^4 + \log\ 6.55]$$

$$= \frac{1}{16} [4 + 0.8162] = 0.3010$$

$$a = \pm\ 2$$

$$[a^{16}b^{16}] = 6.55 \times 10^4 = (ab)^{16}$$

$$\log |ab| = \frac{1}{16} \log 6.55 \times 10^4$$

$$\log b = \frac{1}{16}[4.8162] - 0.301 = 0$$

$$b = \pm 1 \text{ the roots are then } -2,\ -1.$$

COMPLEX ROOTS

Suppose the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_n = 0 \tag{139}$$

is of the fifth degree and has for its Encke roots a, b, c, which are real, and $r\epsilon^{\iota\varphi}$, $r\epsilon^{-\iota\varphi}$, which are imaginary

Suppose $|a| > r > |b| > |c|$.

Equation (139) can be written

$$(x + a)(x + r\epsilon^{\iota\varphi})(x + r\epsilon^{-\iota\varphi})(x + b)(x + c) = 0 \tag{140}$$

and an equation whose roots are the m^{th} power of the roots of the given equation (Encke roots) is

$$(x + a^m)(x + r^m\epsilon^{\iota m\varphi})(x + r^m\epsilon^{-\iota m\varphi})(x + b^m)(x + c^m) = 0 \tag{141}$$

Let us multiply these factors:

$$\begin{aligned}&[x^3 + (a^m + b^m + c^m)\, x^2 + (a^mb^m + a^mc^m + b^mc^m)x + a^mb^mc^m]\\ &\qquad [x^2 + r^m(\epsilon^{\iota m\varphi} + \epsilon^{-\iota m\varphi})x + r^{2m}] = 0\end{aligned} \tag{142}$$

Note that

$$r^m(\epsilon^{\iota m\varphi} + \epsilon^{-\iota m\varphi}) = 2r^m \cos m\varphi$$

multiplying (142),

$$\left.\begin{aligned}&x^5 + [(a^m + b^m + c^m) + 2r^m \cos m\varphi]\, x^4\\ &+ \{[(ab)^m + (ac)^m + (bc)^m] + (a^m + b^m + c^m)2r^m\\ &\qquad\qquad \cos m\varphi + r^{2m}\}\, x^3\\ &+ \{(abc)^m + [(ab)^m + (ac)^m + (bc)^m]2r^m \cos m\varphi +\\ &\qquad\qquad r^{2m}(a^m + b^m + c^m)\}x^2\\ &+ \{(abc)^m\, 2r^m \cos m\varphi + r^{2m}\, [(ab)^m + (ac)^m + (bc)^m]\}x\\ &\qquad\qquad + r^{2m}(abc)^m = 0\end{aligned}\right\} \tag{143}$$

Now, since $|a| > r > |b| > |c|$, retaining only the dominant part, this reduces to

$$x^5 + a^m x^4 + a^m 2r^m \cos m\theta x^2 + a^m r^{2m} x^2 + a^m b^m r^{2m} x \\ + (abc)^m r^{2m} = 0 \quad (144)$$

Suppose also, that $r > |a| > |b| > |c|$, retaining dominant part, we arrive at

$$x^5 + 2r^m \cos m\epsilon \cdot x^4 + r^{2m} x^3 + r^{2m} a^m x^2 + a^m b^m r^{2m} x + (abc)^m r^{2m} = 0$$

We see that a pair of complex roots introduces a cosine factor whose presence will be indicated by a fluctuation in sign of one coefficient (which contains the cosine factor) as the root-squaring process is carried out. If in our equation the coefficient of x^{n-1} fluctuates in sign, then there will be a complex root whose absolute value $r > |a| > |b| > |c|$. Also, r always occurs to the $2m$ power except when multiplied by the $\cos \varphi$, to which it belongs We are now able to write down the algebraical equation whose roots (Encke) are some high power m of the roots (Encke) of the given equation and which contains a pair of complex roots.

If the equation contains two pairs of complex roots, there is no added difficulty; for instance, suppose the sixth-degree equation having two pairs of complex roots such that $|a| > r > r_1 > |b|$ remembering that r and r occur to the $2m$ power, then we can write immediately:

$$x^6 + a^m x^5 + 2a^m r^m \cos m\epsilon \cdot x^4 + a^m r^{2m} x^3 + 2a^m r^{2m} r_1{}^m \cos m\epsilon \cdot x^2 + \\ a^m r^{2m} r_1{}^{2m} x + a^m r^{2m} r_1{}^{2m} b^m = 0 \quad (145)$$

or suppose

$$|a| > r > |b| > r_1$$

then

$$x^6 + a^m x^5 + 2r^m a^m \cos m\epsilon \cdot x^4 + a^m r^{2m} x^3 + a^m r^{2m} b^m x^2 \\ + 2a^m r^{2m} b^m r_1{}^m \cos m\epsilon \cdot x + a^m b^m r^{2m} r_1{}^{2m} = 0 \quad (146)$$

We are now able to find the real roots and the absolute value of the complex roots, but we are also interested in the real and imaginary parts of the complex roots.

Suppose that one pair of complex roots and three real rootsl a_0, b_0, c_0, of the preceding fifth-degree equation are the actua, roots—not Encke roots. $(u + iv)$, $(u - iv)$ are the actual complex roots.

If the original equation is written

$$x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = 0 \quad (147)$$

then the sum of the roots satisfies the relation

$$2u + a_0 + b_0 + c_0 = -a_1 \qquad (148)$$

from which u may be found. When r is found, we may find v:

$$v = \sqrt{r^2 - u^2}$$

The two complex roots are thus determined, a_0 b_0 c_0 having been determined by the method previously discussed for real roots.

CUBIC EQUATION HAVING ONE PAIR OF COMPLEX ROOTS

$$x^3 - 15x - 126 = 0 \qquad (149)$$

NOTE.—1.26^2 denotes 1.26×10^2
2.25^2 denotes 2.25×10^2, etc.

	x^3	a_1x^2	a_2x	a_3
p	1	0	-15	-1.26^{2}
	1	0	2.25^{2}	1.588^{4}
		30	0	
p^2	1	30	2.25^{2}	1.588^{4}
	1	9.00^{2}	5.063^{4}	2.528^{8}
		-4.50^{2}	-95.28^{4}	0
p^4	1	4.50^{2}	-90.22^{4}	2.53^{8}
	1	$2\ 02^{5}$	8.1^{11}	6.4^{16}
		18.044^{5}	-2.28^{11}	
p^8	1	20.06^{5}	5.82^{11}	6.40^{16}
	1	4.02^{12}	3.38^{23}	4.09^{33}
		-1.164^{12}	-2.57^{23}	
p^{16}	1	2.86^{12}	$.81^{23}$	4.09^{33}
	1	8.818^{24}	6.54^{45}	1.67^{67}
		-0.162^{24}	-23.4^{45}	
p^{32}	1	8.02^{24}	-16.86^{45}	1.67^{67}
		6.41^{49}	2.84^{92}	2.79^{134}
		0.00337^{49}	Stop as soon as doubled products become negligible.	
p^{64}		6.413^{49}		

$\therefore$ Stop at p^{32}. NOTE.—We may use a slide rule for the root-squaring process, but from now on it is best to resort to longhand, whence our results will be correct to the third digit. If we continue to use the slide rule, our results will be correct only to the second digit. This is because the value of the roots depends largely upon the exponent to which the (10) is raised, as, for instance, the coefficient of x^2 is 8.02×10^{24}. When we take logarithms, the value of the root is determined largely by the 24. By using the slide rule in the root-squaring process, we can arrive at the correct exponent.

We note that the coefficient of x oscillates in sign as we carry out the root-squaring process. This indicates a pair of complex roots. We may, therefore, write down our p^m equation. Also, we know that $|a| > r$, since the cosine term is in next to the last term.

$$x^3 + a^m x^2 + 2a^m r^m \cos m\theta \; x + a^m r^{2m} = 0 \qquad (150)$$

$$p^{32} = x^3 + 8.02 \times 10^{24} x^2 - 16.86 \times 10^{45} x + 1.67 \times 10^{67} = 0 \qquad (151)$$

Now continuing our work in longhand, comparing (149) and (150), and noting that $m = 32$,

$$\begin{aligned} \log a^{32} &= 24.9042 \\ \log |a| &= 0.7782 \\ a &= 6.00 \end{aligned}$$

It is necessary to use four-place logarithm tables to find the root correctly to three places.

$$\log (ar^2)^{32} = 67.2227$$

$$\log ar^2 = 2.1007 \qquad ar^2 = 126.1$$

$$r^2 = \frac{126.1}{6} = 21.01$$

$$r = 4.59$$

$$\begin{aligned} 2u + a &= -a_1 \text{ from (148)} \\ 2u &= -6 \\ u &= -3 \quad v = \sqrt{r^2 - u^2} = \sqrt{21.01 - 9} = \sqrt{12} = 2\sqrt{3} \end{aligned}$$

The roots are

$$\therefore u \pm iv = -3 \pm 2\sqrt{3}\, i$$

$$a = 6$$

SEVENTH DEGREE EQUATION HAVING TWO PAIRS OF COMPLEX ROOTS

To find $u, v; u', v'$

Consider a seventh-degree equation having the roots

$$a_0, b_0, c_0, \qquad u \pm iv, u' \pm iv'$$

The equation may be written

$$(x - a_0)(x - b_0)(x - c_0)(x - u + iv)(x - u - iv)(x - u' + iv')(x - u' - iv') = 0$$

i.e.,

$$(x - a_0)(x - b_0)(x - c_0)(x^2 - 2xu + u^2 + v^2)(x^2 - 2u'x + u'^2 + v'^2) = 0$$

$$(x - a_0)(x - b_0)(x - c_0)(x^2 - 2xu + r^2)(x^2 - 2u'x + r'^2) = 0$$

$$[x^3 - (a_0 + b_0 + c_0)x^2 + (a_0b_0 + a_0c_0 + b_0c_0)x - a_0b_0c_0][x^2 - 2xu + r^2][x^2 - 2u'x + r'^2] = 0$$

Multiplying out, we have, for the coefficient of x,

$$(a_0b_0 + a_0c_0 + b_0c_0)r^2r'^2 + 2ua_0b_0c_0r'^2 + 2u'a_0b_0c_0r^2$$

or

$$r^2r'^2 (a_0b_0 + a_0c_0 + b_0c_0) + 2a_0b_0c_0(ur'^2 + u'r^2) = a_6 \qquad (152)$$

of which all is known except u and u'.

We also have, of course, the equation

$$a_0 + b_0 + c_0 + 2u + 2u' = -a_1 \qquad (153)$$

from our original equation

$$x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7 = 0$$

From (152) and (153), we can find u and u'. Then v and v' can be found as previously discussed.

SEVENTH-DEGREE EQUATION HAVING THREE PAIRS OF COMPLEX ROOTS

Write equation in form

$$(x - a)(x - u_1 - jv_1)(x - u_1 + jv_1)(x - u_2 - jv_2)(x - u_2 + jv_2) \cdots = 0$$

$$(x - a)(x^2 - 2u_1x + r_1^2)(x^2 - 2u_2x + r_2^2)(x^2 - 2u_3x + r_3^2) = 0$$

Multiplying out:

$$\begin{aligned}(x^2 - 2u_1x + r_1^2)(x^2 - 2u_2x + r_2^2) = [x^4 &- 2(u_1 + u_2)x^3 \\ &+ (r_1^2 + r_2^2 + 4u_1u_2)x^2 - 2(u_2r_1^2 + u_1r_2^2)x + r_1^2r_2^2]\end{aligned}$$

Multiplying this expression by $(x^2 - 2u_3x + r_3^2)$, we have

$$\begin{aligned}x^6 - 2(u_1 + u_2 + u_3)x^5 + (r_1^2 + r_2^2 + r_3^2 + 4u_1u_2 + 4u_1u_3 \\ + 4u_2u_3)x^4 \\ -2(u_2r_1^2 + u_1r_2^2 + u_3r_1^2 + u_3r_2^2 + u_1r_3^2 + u_2r_3^2 + 4u_1u_2u_3)x^3 \\ +(r_1^2r_2^2 + r_1^2r_3^2 + r_3^2r_2^2 + 4u_3u_2r_1^2 + 4u_3u_1r_2^2 + 4u_1u_2r_3^2)x^2 \\ -2(u_3r_1^2r_2^2 + r_3^2r_1^2u_2 + u_1r_2^2r_3^2)x + r_1^2r_2^2r_3^2 = 0\end{aligned}$$

Now multiply by $(x - a)$ and find the coefficient of x to be

$$(r_1^2r_2^2r_3^2 + 2ar_1^2r_2^2u_3 + 2ar_1^2r_3^2u_2 + 2ar_2^2r_3^2u_1) = a_6$$

Also, the coefficient of x^2 is found to be

$$\begin{aligned}-2r_1^2r_2^2u_3 - 2r_3^2r_1^2u_2 - 2r_2^2r_3^2u_1 - a(r_1^2r_2^2 + r_1^2r_3^2 + r_3^2r_2^2) \\ -4a(r_1^2u_2u_3 + r_2^2u_1u_3 + r_3^2u_1u_2) = -a_5\end{aligned}$$

And, likewise, the coefficient of x^5, which is somewhat simpler and is the equation used in the illustrative problem, is found to be

$$\begin{aligned}(r_1^2 + r_2^2 + r_3^2) + 4(u_1u_2 + u_2u_3 + u_1u_3) + 2a(u_1 + u_2 + u_3) \\ = a_2\end{aligned}$$

APPLICATION TO A SEVENTH-DEGREE EQUATION HAVING THREE PAIRS OF COMPLEX ROOTS

$$x^7 - 0.5x^4 + 2x^3 - 2.5 = 0$$

$$x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7$$

	x^7	x^6	x^5	x^4	x^3	x^2	x	C
p	1	0	0	-0.5	2	0	0	-2.5
	1	0	0	0.25	4	0	0	6.25
			4	0	0	2.5	0	
p^2	1	0	4	0.25	4	2.5	0	6.25
	1	0	16	0.0625	16	6.25	0	3.91^1
		-8	8	-32	-1.25	3.13	-31.25	
p^4	1	-8	24	-31.94	14.75	9.38	-31.25	3.91^1
	1	64	5.76^2	1.02^3	2.19^2	8.80^1	9.80^2	1.53^2
		-48	-5.11^2	-0.709^3	5.99^2	92.3^1	-7.34^2	
			0.295^2	-0.150^3	-15.0^2	-250^1		
				0.0625^1	6.25			
p^8	1	16	0.945^2	0.223^3	-0.57^1	-1.49^3	2.46^1	1.53^1
	1	2.56^2	8.83^1	4.93^4	3.25^1	2.22^6	6.05^4	2.34^6
		-1.88^2	-7.14^8	1.08^1	665^1	0.0276^6	455^4	
			-0.114^7	-4.76^1	46.2^3	0.68^6		
				-0.049^1	-49^1			
p^{16}	1	6.8^1	1.58^1	1.20^1	6.65^1	2.93^6	4.61^6	2.34^6
	1	4.62^3	2.5^6	1.44^8	4.42^{11}	8.59^{10}	2.13^{13}	5.48^{12}
		3.16^3	-1.63^6	-21^8	-0.704^{11}	-6.14^{11}	-1.37^{12}	
			1.33^6	3.98^8	-0.146^{11}	0.0562^{12}		
				-0.0924^8	-0.0032^{11}			
p^{32}	1	1.46^1	2.20^6	-1.57^9	3.86^{11}	2.506^{12}	7.6^{12}	5.48^{12}
	1	2.13_6	4.84^{12}	2.46^{18}	1.49^{23}	6.3^{24}	5.78^{25}	3.00^{25}
		-4.40^6	4.59^{12}	-1.70^{18}	0.0787^{23}	-5.87^{24}	-2.75^{25}	
			0.772^{12}	0.007^{18}	0.0003^{23}	-0.0172^{24}		
				-0.00002^{18}				
p^{64}	1	-2.27^6	1.02^{13}	7.67^{17}	1.57^{23}	4.13^{24}	3.03^{25}	3.00^{25}
	1	5.15^{12}	1.04^{26}	5.88^{35}	2.46^{46}	1.71^{47}	9.18^{50}	9.00^{50}
		-20.4^{12}	0.035^{26}	-3×1.7^{35}	0.00006^{46}	-95.2^{47}	-0.248^{50}	
				-0.000019^{41}	-0.00006^{46}			
p^{128}	1	-1.53^{13}	1.075^{26}	-2.58^{36}	2.46^{46}	-9.35^{48}	8.93^{50}	9.00^{50}
	1	2.34^{26}	1.17^{52}	6.66^{72}	6.05^{92}	8.74^{97}	7.97^{101}	8.1^{101}
		-2.15^{26}	-0.008^{52}	-5.37^{72}	0000	-4.39^{97}	0.168^{101}	
p^{256}	1	1.9^{25}	1.16^{52}	1.36^{77}	6.05^{92}	4.35^{97}	8.14^{101}	8.1^{101}
	1	3.61^{50}	1.35^{104}	1.85^{144}	3.66^{185}	1.89^{195}	6.63^{203}	6.56^{203}
		-232^{50}	000	-14.1^{144}	000	0.985^{195}	000	
p^{512}	Coefficients are practically squares of p^{256}							

∴ **Stop at $m = 256$**

Let us rewrite the p equations for comparison:

	x^7	x^6	x^5	x^4	x^3	x^2	x	c
p	1	0	0	-0.500	2.00	0	0	-2.50
p^2	1	0	4	0.250	4.00	2.50	0	6.25
p^4	1	-8.00	24.0	-31.94	14.75	9.38	-31.25	3.91^{1}
p^8	1	16.0	9.45^{1}	2.23^{2}	-5.7^{1}	-1.49^{3}	2.46^{2}	1.53^{3}
p^{16}	1	6.80^{1}	1.58^{3}	1.20^{4}	6.65^{5}	2.93^{6}	4.61^{6}	2.34^{6}
p^{32}	1	1.46^{3}	2.20^{6}	-1.57^{9}	3.86^{11}	2.51^{12}	7.6^{12}	5.48^{12}
p^{64}	1	-2.27^{6}	1.02^{13}	7.67^{17}	1.57^{23}	4.13^{23}	3.03^{25}	3.00^{25}
p^{128}	1	-1.53^{13}	1.08^{26}	-2.58^{36}	$+2.46^{46}$	-9.35^{48}	8.93^{50}	9.00^{50}
p^{256}	1	1.90^{25}	1.16^{52}	1.36^{72}	6.05^{92}	4.35^{97}	8.14^{101}	8.10^{101}

$$\begin{aligned} p^{256} = x^7 + 1.90^{25}x^6 + 1.16^{52}x^5 + 6.05^{92}x^3 + 4.35^{97} \\ x^2 + 8.14^{101}x + 8.10^{101} = 0 \qquad (154) \end{aligned}$$

In the above tabulation the coefficients of x^6, x^4, x^2, oscillate in sign, indicating three pairs of complex roots, $r_1\epsilon^{\pm j\theta_1}, r_2\epsilon^{\pm j\theta_2}, r_3\epsilon^{\pm j\theta_3}$. The remaining root ($a$) is real and the smallest numerically. We can write down the p^m equation as follows:

$$\begin{aligned} x^7 + 2r_1{}^m\cos\ m\theta_1x^6 + r_1{}^{2m}x^5 + 2r_1{}^{2m}r_2{}^m\cos\ m\theta_2x^4 \\ + r_1{}^{2m}r_2{}^{2m}x^3 + 2r_1{}^{2m}r_2{}^{2m}r_3{}^m\cos\ m\theta_3x^2 \\ + r_1{}^{2m}r_2{}^{2m}r_3{}^{2m}x + r_1{}^{2m}r_2{}^{2m}r_3{}^{2m}a^m = 0 \qquad (155) \end{aligned}$$

Using equations (154) and (155) for comparison:

$$m = 256$$

$\log(r_1{}^2)^m = 52.0645 \qquad \log r_1{}^2 = 0.2034$

$r_1{}^2 = 1.597$

$\log(r_1{}^2r_2{}^2)^m = 92.7818 \qquad \log r_1{}^2r_2{}^2 = 0.3624$

$r_1{}^2r_2{}^2 = 2.304$

$\log(r_1{}^2r_2{}^2r_3{}^2)^m = 101.9106 \qquad \log r_1{}^2r_2{}^2r_3{}^2 = 0.3981$

$r_1{}^2r_2{}^2r_3{}^2 = 2.501$

$\log(r_1{}^2r_2{}^2r_3{}^2)^ma^m = 101.9085 \qquad \log r_1{}^2r_2{}^2r_3{}^2a = 0.3981$

$r_1{}^2r_2{}^2r_3{}^2a = 2.501$

$$a = \frac{2.501}{2.501} = 1.000$$

$\log r_1 = 0.1017 \qquad r_1 = 1.264 \qquad r_1 = 1.264 \qquad r^2{}_1 = 1.597$

$\log r_1r_2 = 0.1812 \qquad r_1r_2 = 1.528 \qquad r_2 = 1.209 \qquad r^2{}_2 = 1.443$

$\log r_1r_2r_3 = 0.1991 \qquad r_1r_2r_3 = 1.582 \qquad r_3 = 1.035 \qquad r^2{}_3 = 1.086$

It remains to find $u_1, u_2, u_3, v_1, v_2, v_3$. The complex roots may be written $u_1 \pm iv_1$, $u_2 \pm iv_2$, $u_3 \pm iv_3$.

Now the sum of the roots is equal to $-a_1 = 0$ = coefficient of x^6 *i.e.*,

$$u_1 + u_2 + u_3 = -\frac{a}{2} \tag{156}$$

Coefficient of x,

$$r_1^2r_2^2r_3^2 + 2ar_1^2r_2^2u_3 + 2ar_1^2r_3^2u_2 + 2ar_2^2r_3^2u_1 = a_6 \tag{157}$$

Coefficient of x^5,

$$(r_1^2 + r_2^2 + r_3^2) + 4u_1u_2 + 4u_1u_3 + 4u_2u_3 + 2a(u_1 + u_2 + u_3) = a_2 \tag{158}$$

Equation (156) becomes $u_1 + u_2 + u_3 = -0.5$. (157) becomes $2.501 + 4.608u_3 + 3.469u_2 + 3.134u_1 = 0$. (158) becomes $4.126 + 4(u_1u_2 + u_1u_3 + u_2u_3) - 1 = 0$

From (156) and (157) eliminate u_3 and u_2 in terms of u_1 by means of the following two equations

$$\begin{Bmatrix} u_2 = & 0.1729 - 1.294u_1 \\ u_3 = & -0.6729 + 0.294u_1 \end{Bmatrix}$$

Substitute in equation (158) and get

$$\begin{array}{rrr} 0.7815 & +0.1729u_1 & -1.294u_1^2 \\ -0.1163 & -0.6729u_1 & +0.294u_1^2 \\ & +0.0508u_1 & -0.3804u_1^2 \\ & +0.8707u_1 & \\ \hline 0.6652 & +0.4215u_1 & -1.38u_1^2 = 0 \end{array}$$

which is quadratic

$$u_1 = \frac{0.4215 \pm \sqrt{0.1776 + 4 \times 1.38 \times 0.6652}}{2.76}$$

$$u_1 = 0.864$$

The other value of u_1 does not satisfy the original equation.

$$v_1 = \sqrt{1.597 - 0.7465} = \sqrt{0.85} = 0.922$$

$\therefore u_1 \pm jv_1 = 0.864 \pm j0.922$, correct to three places.

$$u_2 = -.946,\ u_3 = -0.419,\ v_2 = 0.75,\ v_3 = 0.954$$

$\therefore$ the roots are $a = +1$

$$\begin{aligned} u_1 \pm jv_1 &= 0.864 \pm 0.922j \\ u_2 \pm jv_2 &= -0.946 \pm 0.750j \\ u_3 \pm jv_3 &= -0.419 \pm 0.954j \end{aligned}$$

correct to three places

COINCIDENT ROOTS

Suppose the roots a, b, b', c, d . . . are Encke roots. b and b' are coincident. And $|a| > |b| > |c| > |d| > \cdots$ We can write the equation

$$(x + a)(x + b)(x + b')(x + c)(x + d) \cdots = 0$$

The equation whose Encke roots are the m^{th} power of the Encke roots of the given equation is

$$(x + a^m)(x + b^m)(x + b'^m)(x + c^m)(x + d^m) \cdots = 0$$

which may be written

$$x^n + (a^m + b^m + b'^m + c^m + d^m \cdots)x^{n-1} + [(ab)^m + (ab')^m + (bb')^m + (bc)^m + (b'c)^m + (b'd)^m + (bd)^m + (ac)^m + (ad)^m + \cdots]x^{n-2} + [(abc)^m + (abb')^m + (abd)^m + \cdots]x^{n-3} + [(abb'c)^m + \cdots]x^{n-1} + \cdots (abb'cd \cdots)^m = 0$$

Retaining only the dominant terms, we have

$$x^n + a^m x^{n-1} + [2a^m b^m]x^{n-2} + a^m b^{2m} x^{n-3} + a^m b^{2m} c^m x^{n-4} + \cdots = 0$$

Notice the coefficient of x^{n-2}. If m is doubled, the coefficient obtained will not be approximately the square of the corresponding coefficient as has previously been true, but will be approximately one-half the square of the corresponding coefficient. The coefficient of x^{n-2} is $(2a^m b^m)$. (Squaring we get $4a^{2m}b^{2m}$; doubling m we get $2a^{2m}b^{2m}$.)

Thus, we can detect the presence of the coincident root.

Illustrative Example:

$$x^3 + 7x^2 + 16x + 12 = 0$$

The successive equations are

p^2 $x^3 + 17x^2 + 88x + 144 = 0$
p^4 $x^3 + 113x^2 + 2{,}848x + 20{,}736 = 0$
p^8 $x^3 + 707x^2 + 3.43 \times 10^6 x + 4.3 \times 10^8 = 0$
p^{16} $x^3 + 4.32 \times 10^7 x^2 + 0.565 \times 10^{13} x + 1.85 \times 10^{17} = 0$
p^{32} $x^3 + 1.85 \times 10^{15} x^2 + 1.65 \times 10^{25} x + 3.42 \times 10^{34} = 0$

It is evident that the coefficients of p^{64} would be the squares of the corresponding coefficients of p^{32} except in the case of the coefficient of x. This indicates the presence of coincident roots.

$$x^3 + a^m x^2 + 2a^m b^m x + a^m b^{2m} = 0 \qquad |a| > |b|$$

$\log a^{32} = 15.2679 \qquad \log |a| = 0.4771 \qquad |a| = 3.00$

$\log a^{32}b^{64} = 34.5339 \qquad \log |b| = 0.3010 \qquad |b| = 2.00 = b'$

The actual roots are -3, -2, -2.

PURE IMAGINARY ROOTS

To be definite, take a cubic equation having one real root, a, and one pair of imaginary roots, $r\epsilon^{+j\theta}$, $r\epsilon^{-j\theta}$.

$$(x + a)(x + r\epsilon^{j\theta})(x + r\epsilon^{-j\theta}) = 0 \qquad (159)$$

is the equation whose roots are the given roots with their signs reversed.

$$(x + a^m)(x + r^m\epsilon^{jm\theta})(x + r^m\epsilon^{-jm\theta}) = 0 \qquad (160)$$

has Encke roots equal to the m^{th} power of the Encke roots of the preceding equation. Expanding, we have

$$x^3 + (a^m + 2r^m \cos m\theta)x^2 + (2a^m r^m \cos m\theta + r^{2m})x + a^m r^{2m} = 0 \qquad (161)$$

Now let $\theta = \pi/2$

$$x^3 + \left(a^m + 2r^m \cos m\frac{\pi}{2}\right)x^2 + \left(2a^m r^m \cos m\frac{\pi}{2} + r^{2m}\right)x + a^m r^{2m} = 0 \qquad (162)$$

Suppose $|a| > r$ and retain the dominant part

$$p^m \text{ is } x^3 + a^m x^2 + 2a^m r^m \cos \frac{m\pi}{2} x + a^m r^{2m} = 0 \qquad (163)$$

When $m = 2$; *i.e.*, finding p^2, we have $\cos \pi$, which is (-1). Thus, the coefficient of x when m goes from p to p^2 will change sign, but when m goes from 2 to 4, or 8 to 16, etc., we have $\cos 2\pi$, $\cos 4\pi$, $\cos 8\pi$, which is always positive.

Also, note that doubling m gives for the new coefficient $2a^{2m}r^{2m} \cos \pi m = 2a^{2m}r^{2m}$. But the square of the original coefficient is

$$4a^{2m}r^{2m}\left(\cos \frac{\pi m}{2}\right)^2 = 4a^{2m}r^{2m}$$

For example, take $m = 2$ in equation (162):

$$x^3 + (a^2 + 2r^2 \cos \pi)x^2 + (2a^2r^2 \cos \pi + r^4)x + a^2r^4 = 0 \qquad (164)$$

Now, if the real root, is larger than r, it will be much larger when m reaches 32 or 64 or may be so when $m = 2$.

Retaining the dominant part and doubling m, we have for the coefficient of x, $2a^4r^4 \cos 2\pi$. But the square of the coefficient of x is $4a^4r^4 \cos^2\pi$, which is twice that obtained by doubling (m). Therefore, to detect the roots $\pm jv$:

1. There will be one change of sign of the affected coefficient, namely when m goes from 1 to 2, after which the coefficient of that power of x will always be of the same sign—*i.e.*, when m is going from 2 to 4 to 8, etc. · · ·

2. As the root-squaring process is carried out and $|a|$ becomes enormously larger than $|r|$, the affected coefficient will be approximately one-half the square of the preceding coefficient.

Application to a Circuit with Concentrated Constants.—A voltage E is suddenly impressed upon the following circuit at time $t = 0$. Find the current in the coil of inductance L at time t after the switch is closed

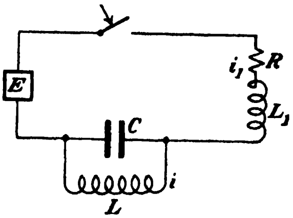

FIG. 53

The total impedance operator is

$$Z = R + pL_1 + \frac{pL}{p^2CL + 1} = \frac{(R + pL_1)\,(p^2CL + 1) + pL}{p^2CL + 1}$$

giving the current in the resistance R

$$i_1 = \frac{E1}{Z} = \frac{E(p^2CL + 1)1}{(R + pL_1)(p^2CL + 1) + pL}$$

Now, if i is the current in coil of inductance L

$$\frac{i}{i_1} = \frac{\text{coil admittance}}{\text{branch admittance}} = \frac{1/pL}{\dfrac{1}{pL} + pC}$$

$$i = \frac{i_1}{1 + p^2CL}$$

Thus,

$$i = \frac{E\mathbf{1}}{CLL_1p^3 + RCLp^2 + (L + L_1)p + R}$$

Suppose $E = 100$ volts $\quad L = L_1 = 2$ henries

$R = 400$ ohms

$C = 1$ farad

Then

$$i = \frac{100\,\mathbf{1}}{4p^3 + 800p^2 + 4p + 400}$$

Solving by use of the expansion theorem,

$$Y(p) = 100$$

$$Z(p) = 4p^3 + 800p^2 + 4p + 400 = 4(p^3 + 200p^2 + p + 100)$$

$$p\frac{dZ}{dp} = 4(3p^3 + 400p^2 + p)$$

$$\frac{Y_{(0)}}{Z_{(0)}} = \frac{E}{R} = \frac{100}{400} = \frac{1}{4}$$

$$i = \frac{Y_{(0)}}{Z_{(0)}} + \sum_{p_1p_2p_3} \frac{Y_{(p)}}{p\frac{dZ}{dp}}\epsilon^{pt}$$

where the summation is for the roots of $Z(p) = 0$. Applying the root-squaring method:

	p^3	p^2	p	c
x	1	200	1	100
	1	4^4	1	1^4
		-2	-4^4	
x^2	1	3.9998^4	$-\ 3.9999^4$	1^4
	1	15.9984^8	15.9992^8	1^8
		7.9998^4	7.9996^4	
x^4	1	15.9992^8	15.9984^8	1^8
	1	256.9744^{16}	255.9488^{16}	1^{16}
x^8	1	256.9744^{16}	255.9488^{16}	1^{16}

There is no change from $(x^4)^2$ to x^8, so we can stop with x^4. The coefficient of p oscillates, indicating a pair of complex roots.

The equation whose roots with sign reversed are the m^{th} power of the roots with sign reversed of $Z(p)$, is

$$p^3 + a^m p^2 + (2a^m r^m \cos m\theta)p + a^m r^{2m} = 0$$

$$\begin{aligned}
a^m &= 15.9992 \times 10^8 \\
m \log a &= \log 15.9992 \times 10^8 \\
\text{since } m &= 4 \\
\log a &= \frac{1}{4} \times 9.204098 \\
\log a &= 2.3010245 \\
a &= 199.9975 \\
p_1 &= -199.9975
\end{aligned}$$

$$\begin{aligned}
(ar^2)^m &= 10^8 \\
4 \log ar^2 &= 8 \\
\log ar^2 &= 2.000 \\
ar^2 &= 100 \\
r^2 &= \frac{100}{200} = \frac{1}{2} \\
r &= \sqrt{1/2}
\end{aligned}$$

The easiest way to find the other roots is to divide the original equation by $(p + 199.9975)$ and solve the resulting quadratic

$$p^2 + 0.0025p + 0.5p = 0$$

Thus,

$$\begin{aligned}
p_2 &= -0.00125 + j\sqrt{1/2} \\
p_3 &= -0.00125 - j\sqrt{1/2}
\end{aligned}$$

By the root-squaring method, we proceed as follows:

$$\text{The roots are } \left|\begin{array}{l} u + iv \\ u - iv \\ a \end{array}\right.$$

and since the sum of the roots is equal to minus the coefficient of p^2,

$$\begin{aligned}
2u + a &= -200 \\
u &= -0.00125 \\
r = \sqrt{1/2} \quad v &= \sqrt{r^2 - u^2}
\end{aligned}$$

The roots are then

$p_1 = -199.9975$, $p_2 = -0.00125 + 0.707j$,
$p_3 = -0.00125 - 0.707j$

$$\left[\frac{Y(p)}{p\dfrac{dZ}{dp}} e^{pt}\right]_{p=p'_1} = -3.125 \times 10^{-6} e^{-199.9975t}$$

The other two roots $p_2 \cong 0.707\underline{|90^\circ 06'}$, $p_3 \cong 0.707\underline{|269^\circ 54'}$ give

$$\frac{100e^{(-0.00125+0.707j)t}}{4(1.065\,\underline{|270^\circ 18'} + 200\underline{|180^\circ 12'} + 0.707\underline{|90^\circ 06'}} + \frac{100e^{(-0.00125-.707j)t}}{4(1.065\underline{|89^\circ 42'} + 200\underline{|179^\circ 48'} + 0.707\underline{|269^\circ 54'})}$$

which become on being rationalized and simplified

$$= -\frac{e^{-0.0125t}}{4}\left\{\left[\frac{e^{0.707jt} - e^{-0.707jt}}{2}\right] + \frac{0.042}{8}\left[\frac{e^{0.707jt} - e^{-0.0707jt}}{2j}\right\}\right]$$

$$= -0.25e^{-0.00125t}\cos(0.707t - \theta) \text{ where } \tan\theta = \frac{0.042}{8}$$

$$\theta = 0^\circ 18'$$

Therefore, the answer is $I_L = 0.25 - 3.125 \times 10^{-6}e^{-199.9975} - 0.25e^{-0.00125t}\cos(0.707t - 0^\circ 18')$.

Application to a Circuit with Distributed and Concentrated Constants.—A voltage E is suddenly impressed upon an "ideal" cable of length l through a concentrated inductance L_0. Find the voltage V_0 at the beginning of the cable.

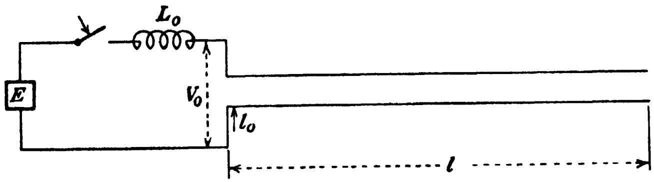

FIG. 54

$$V_0 = E\mathbf{1} - i_0pL_0;\ i_0 = \frac{YV_0\sinh nl}{n\cosh nl}\ \therefore V_0 + \frac{V_0pL_0Y\sinh nl}{n\cosh nl} = E\mathbf{1},$$

$$V_0 = \frac{E\mathbf{1}}{1 + \dfrac{pL_0Y\sinh nl}{n\cosh nl}};\ V_0 = \frac{nE\cosh nl}{n\cosh nl + pL_0Y\sinh nl}\mathbf{1}$$

$$V_0 = \frac{nE\cosh nl}{n\cosh nl + p^2CL_0\sinh nl}\mathbf{1} \tag{165}$$

since $n^2 = pCR$ and $Y = pC$

Transforming to circular functions by substituting $n^2 = -m^2$, $n = \pm jm$,

$$V_0 = \frac{\pm jm\, E \cosh\,(\pm jml)}{\pm jm \cosh\,(\pm jml) + p^2CL_0 \sinh\,(\pm jml)} 1 = \frac{Em \cos ml}{m \cos ml + p^2CL_0 \sin ml} 1 \tag{166}$$

$$\begin{aligned} Y(p) &= m \cos ml \\ Z(p) &= m \cos ml + p^2CL_0 \sin ml \end{aligned} \tag{167}$$

$\dfrac{Y(0)}{Z(0)}$ is the limit of $\left[\dfrac{m \cos ml}{m \cos ml + p^2CL_0 \sin ml} E1\right]$ as m approaches zero

$$= \lim \left[\frac{\left(\dfrac{-CR \cos ml}{2m} + \dfrac{CRl \sin ml}{2}\right)}{\dfrac{-CR \cos ml}{2m} + \dfrac{CRl \sin ml}{2} - \dfrac{p^2C^2RL_0l \cos ml}{2m} + 2pCL_0 \sin ml} E1\right]$$

as m approaches zero,

$$= \lim_{\substack{m\to 0\\ p\to 0}} \left[\frac{(-CR \cos ml + mCRl \sin ml)E1}{-CR \cos ml + CRml \sin ml + p^2CRlL_0 \cos ml + 4mpCL_0 \sin ml}\right]$$

$$= E1$$

$$\frac{dZ}{dp} = (\cos ml - ml \sin ml + p^2CL_0l \cos ml)\frac{dm}{dp} + 2pCL_0 \sin ml$$

$$n^2 = pCR = -m^2 \qquad -2m\frac{dm}{dp} = CR \qquad \frac{dm}{dp} = -\frac{CR}{2m}$$

$$\frac{dZ}{dp} = \frac{-CR \cos ml}{2m} + \frac{CRl \sin ml}{2} - \frac{L_0p^2C^2Rl \cos ml}{2m} + 2pCL_0 \sin ml$$

$$p\frac{dZ}{dp} = \frac{-pCR \cos ml}{2m} + \frac{pCRl \sin ml}{2} - \frac{p^3C^2RL_0l \cos ml}{2m} + 2p^2CL_0 \sin ml$$

$$= \frac{m \cos ml}{2} - \frac{m^2l}{2} \sin ml + \frac{p^2CL_0lm}{2} \cos ml + 2p^2CL_0 \sin ml$$

Equating $Z(p)$ to 0

$$m\cos ml = -p^2CL_0\sin ml \tag{168}$$

$$\tan\alpha = \frac{-m}{p^2CL_0} = \frac{-\alpha}{p^2CL_0l} = \frac{-CR^2l^3}{L_0\alpha^3}$$

$$\frac{Y(p)}{p\dfrac{dZ}{dp}} = \frac{1}{\dfrac{1}{2} - \dfrac{ml\tan ml}{2} + \dfrac{p^2CL_0lm}{2m} + \dfrac{2p^2CL_0}{m}\tan ml}$$

$$\text{the denominator} = \frac{1}{2} - \frac{\alpha\tan\alpha}{2} - 2 - \frac{\alpha}{2\tan\alpha}$$

$$= -\frac{3}{2} - \frac{\alpha\tan\alpha}{2} - \frac{\alpha}{2\tan\alpha}$$

$$\frac{Y(p)}{p\dfrac{dZ}{dp}} = \frac{2}{-3 - \alpha\tan\alpha - \dfrac{\alpha}{\tan\alpha}}$$

$$= \frac{2\sin\alpha\cos\alpha}{-3\cos\alpha\sin\alpha + 2\alpha(-\cos^2\alpha - \sin^2\alpha)} = -\frac{2\sin 2\alpha}{2\alpha + 3\sin 2\alpha} \tag{169}$$

and

$$V_0 = E - \sum_{\alpha_0,\alpha_1,\alpha,} \frac{2\sin 2\alpha}{2\alpha + 3\sin 2\alpha}\epsilon^{-\frac{\alpha^2t}{CRl^2}} \tag{170}$$

where

$$\tan\alpha = -\frac{CR^2l^3}{L_0\alpha^3}$$

Substituting numerical values:

$$C = \frac{2}{5\times 10^6}\text{ farads per loop mile}$$

$$R = 2\text{ ohms per loop mile}$$

$$L_0 = 1\text{ henry per loop mile}$$

$$l = 200\text{ miles}$$

equation (168) becomes

$$\tan\alpha = -\frac{12.8}{\alpha^3} \tag{171}$$

In order to apply the root-squaring method to the above transcendental equation, $\tan\alpha$ must be represented by a suitable infinite series which converges fairly rapidly.

$$\tan\alpha = \alpha + 3\alpha^3 + \frac{2}{15}\alpha^5 + \frac{17}{315}\alpha^7 + \frac{62\alpha^9}{2{,}835} + \cdots \tag{172}$$

where $\alpha^2 < \frac{\pi^2}{4}$. But writing equation (171) as $\cot \alpha = \frac{-\alpha^3}{12.8}$, a series can be used representing cot α for larger values of α. Thus,

$$\cot \alpha = -\frac{\alpha^3}{12.8} = \frac{1}{\alpha} - \frac{\alpha}{3} - \frac{\alpha^3}{45} - \frac{2\alpha^5}{945} - \frac{\alpha^7}{4{,}725} - \cdots \quad (173)$$

where $\alpha^2 < \pi^2$. Using this much of the cotangent series for purposes of calculation, we get this equation to solve:

$$0 = 1 - \frac{\alpha^2}{3} + \frac{32.2}{576}\alpha^4 - \frac{2\alpha^6}{945} - \frac{\alpha^8}{4{,}725} \quad (174)$$

Letting $x = \alpha^2$ and simplifying, we have

$$x^4 + 10x^3 - 264x^2 + 1{,}575x - 4{,}725 = 0 \quad (175)$$

to solve by the root-squaring method.

NOTE.— $-2.64^2 = -2.64 \times 10^2$, etc. Tabulating:

	x^4	x^3	x^2	x	c
p	1	1^1	$-\ 2.64^2$	1.575^3	-4.725^3
	1	1^2	7.0^4	2.48^6	2.24^7
		5.28^2	$-\ 3.15^4$	2.495^6	
			$-\ 0.95^4$		
p^2	1	6.28^2	2.9^4	$-\ 1.5^4$	2.24^7
	1	3.95^5	8.4^8	2.25^8	5.02^{14}
		-0.58^5	$+\ 0.188^8$	$-13{,}000^8$	
			$+\ 0.448^8$		
p^4	1	3.37^5	9.036^8	$-\ 1.3^{12}$	5.02^{14}
	1	1.135^{11}	8.16^{17}	1.69^{24}	2.52^{29}
		-0.018^{11}	$+\ 8.76^{17}$	$-\ 0.90^{24}$	
			$+\ 0.01^{17}$		
p^8	1	1.117^{11}	16.93^{17}	0.79^{24}	2.52^{29}
	1	1.245^{22}	2.86^{36}	6.25^{47}	6.34^{58}
		-0.177^{36}	$-\ 8.5^{47}$		
p^{16}	1	1.245^{22}	2.683^{36}	$-\ 2.25^{47}$	6.34^{58}
	1	1.55^{44}	6.69^{72}	5.08^{94}	4.02^{117}
		0.006^{72}	0.006^{72}	-33.76^{94}	
p^{32}	1	1.55^{44}	6.696^{72}	$-\ 2.868^{95}$	4.02^{117}

$$x^4 + a^m x^3 + a^m b^m x^2 + 2a^m b^m r^m \cos m\theta x + a^m b^m r^{2m} = 0$$

where $m = 32$

$$|a| > |b| > r$$

The complex roots are the smallest in absolute value and are determined mainly by the first few terms of the series.

$$32 \log |a| = (\log 1.55) + 44 \qquad \log |a| = \frac{0.19033 + 44}{32} = 1.38095$$

$|a| = 24.04$ and is negative *i.e.* $a = -24.04$

$$\alpha = \pm j\sqrt{24.04} = \pm 4.9j \text{ (does not satisfy)}$$

$$32 \log |ab| = 72 + \log 6.696$$

$$\log |ab| = 2.27581 \quad |ab| = 188.8 \quad b = 7.853 \text{ (positive)}$$

$$32 \log (abr^2) = 117 + \log 4.02 = 3.67513$$

$$abr^2 = 4{,}733$$

$$r^2 = 25.0688$$

$$|x| = r = 5.00 \quad |\alpha| = \pm\sqrt{r} = \pm 2.24$$

$$a + b + 2u = -10$$

$$-16.19 + 2u = -10 \qquad \theta = \cos^{-1}\left(\frac{3.10}{5.00}\right) = 51.7^\circ$$

$$2u = 6.19 \qquad x = 5.00\underline{|51.7^0}$$

$$u = 3.10 \qquad \alpha = \pm\ 2.25\underline{|\pm 26^0}$$

The real roots are determined graphically.

$\alpha_0 =$ $\quad = \pm 2.18\underline{|26^\circ}$ (successive substitutions)

$\alpha_1 = 2.38$ radians $\quad = 136^\circ 21' 50''$

$\alpha_2 = 6.23$ radians $\quad = -3^\circ 02' 21'' \cong 2\pi$

$\alpha_3 = 9.41$ radians $\quad = -0^\circ 51' \quad \cong 3\pi$

$\alpha_4 = 12.57$ radians $\quad = -0^\circ 22' \quad \cong 4\pi$

$\alpha_5 = 15.7$ radians $\quad = \quad \cong 5\pi$

$\alpha_6 = 18.85$ radians $\quad = -0^\circ 07' \quad \cong 6\pi$

$\cdots \qquad \cdots \qquad \cdots$

Substitute the roots in equation (170)

$$V_0 = E[-2.2E\epsilon^{-19.6t} \cos (117t + 13^0) + 1.13\epsilon^{-177t} + 0.017 \epsilon^{-1{,}220t} + 0.00316\epsilon^{-2{,}760t} + 0.001\epsilon^{-4{,}950t} + 0.0002\epsilon^{-11{,}000t} \cdots]$$

At $t = 0$

$$-2.2 \cos 13^0 = -2.2 \times 0.974 = -2.14$$

$$V_0 = -2.14 + 2.14 = 0$$

At $\quad t = \infty$

$$V_0 = E \text{ the battery potential}$$

CHAPTER XXIX

LIST OF OPERATORS AND FORMULAS

Equivalent Operators:

1. $\dfrac{p}{p+\alpha}\boldsymbol{1} = \epsilon^{-\alpha t}\boldsymbol{1}$

2. $\dfrac{p}{p-\alpha}\boldsymbol{1} = \epsilon^{\alpha t}\boldsymbol{1}$

3. $\dfrac{1}{p+\alpha}\boldsymbol{1} = \dfrac{1}{\alpha}(1 - \epsilon^{-\alpha t})\boldsymbol{1}$

4. $\dfrac{1}{p(p+\alpha)}\boldsymbol{1} = \left(\dfrac{t}{\alpha} - \dfrac{1}{\alpha^2} + \dfrac{\epsilon^{-\alpha t}}{\alpha^2}\right)\boldsymbol{1}$

5. $\dfrac{p^2}{(p+\alpha)(p+\beta)}\boldsymbol{1} = \dfrac{1}{\alpha-\beta}[\alpha\epsilon^{-\alpha t} - \beta\epsilon^{-\beta t}]\boldsymbol{1}$

6. $\dfrac{p}{(p+\alpha)(p+\beta)}\boldsymbol{1} = \dfrac{1}{\alpha-\beta}[\epsilon^{-\beta t} - \epsilon^{-\alpha t}]\boldsymbol{1}$

If $\omega_0{}^2 > \alpha^2$, $\omega^2 = \omega_0{}^2 - \alpha^2$ and $\tan\phi = \omega/\alpha$, Then

7. $\dfrac{p^2}{p^2 + 2\alpha p + \omega_0{}^2}\boldsymbol{1} = -\dfrac{\omega_0}{\omega}\epsilon^{-\alpha t}\sin(\omega t - \phi)\boldsymbol{1}$

8. $\dfrac{p}{p^2 + 2\alpha p + \omega_0{}^2}\boldsymbol{1} = \dfrac{\epsilon^{-\alpha t}}{\omega}\sin\omega t\ \boldsymbol{1}$

9. $\dfrac{1}{p^2 + 2\alpha p + \omega_0{}^2}\boldsymbol{1} = \dfrac{1}{\omega_0{}^2}\left[1 - \dfrac{\omega_0}{\omega}\epsilon^{-\alpha t}\sin(\omega t + \phi)\right]\boldsymbol{1}$

If $\alpha^2 > \omega_0{}^2$ and if $(-m)$ and $(-n)$ are the two roots of the equation $p^2 + 2\alpha p + \omega_0{}^2 = 0$, then

10. $\dfrac{p^2}{p^2 + 2\alpha p + \omega_0{}^2}\boldsymbol{1} = \dfrac{1}{n-m}[n\epsilon^{-nt} - m\epsilon^{-mt}]\boldsymbol{1}$

11. $\dfrac{p}{p^2 + 2\alpha p + \omega_0{}^2}\boldsymbol{1} = \dfrac{1}{n-m}[\epsilon^{-mt} - \epsilon^{-nt}]\boldsymbol{1}$

12. $$\frac{1}{p^2 + 2\alpha p + \omega_0{}^2}\mathbf{1} = \left[\frac{1}{\omega_0{}^2} - \frac{1}{n - m}\left(\frac{\epsilon^{-mt}}{m} - \frac{\epsilon^{-nt}}{n}\right)\right]\mathbf{1}$$

If $\alpha^2 = \omega_0{}^2$, then (two equal roots),

13. $$\frac{p^2}{p^2 + 2\alpha p + \omega_0{}^2}\mathbf{1} = \epsilon^{-\alpha t}(1 - \alpha t)\mathbf{1} = \frac{p^2}{(p + \alpha)^2}\mathbf{1}$$

14. $$\frac{p}{p^2 + 2\alpha p + \omega_0{}^2}\mathbf{1} = t\epsilon^{-\alpha t}1 = \frac{p}{(p + \alpha)^2}\mathbf{1}$$

15. $$\frac{1}{p^2 + 2\alpha p + \omega_0{}^2}\mathbf{1} = \frac{1}{\omega_0{}^2}[1 - \epsilon^{-\alpha t}(1 + \alpha t)]\mathbf{1} = \frac{1}{(p + \alpha)^2}\mathbf{1}$$

16. $$\frac{p}{(p - \alpha)^n}\mathbf{1} = \epsilon^{\alpha t}\frac{t^{n-1}}{\underline{|n - 1}}\mathbf{1}$$

17. $$\frac{1}{p^n}\mathbf{1} = \frac{t^n}{\underline{|n}}\mathbf{1}$$

18. $$\text{Sin}\,\omega t\mathbf{1} = \frac{\omega p}{p^2 + \omega^2}\mathbf{1}$$

19. $$\text{Cos}\,\omega t\mathbf{1} = \frac{p^2}{p^2 + \omega^2}\mathbf{1}$$

20. $$\text{Sinh}\,\omega t\mathbf{1} = \frac{\omega p}{p^2 - \omega^2}\mathbf{1}$$

21. $$\text{Cosh}\,\omega t\mathbf{1} = \frac{p^2}{p^2 - \omega^2}\mathbf{1}$$

22. $$\epsilon^{-\beta t}\sin\,\omega t\mathbf{1} = \frac{p\omega}{(p + \beta)^2 + \omega^2}\mathbf{1}$$

23. $$\epsilon^{-\beta t}\cos\,\omega t\mathbf{1} = \frac{p(p + \beta)}{(p + \beta)^2 + \omega^2}\mathbf{1}$$

24. $$\text{Sin}\,(\omega t \pm \phi)\mathbf{1} = \frac{p\omega\cos\phi \pm p^2\sin\phi}{p^2 + \omega^2}\mathbf{1}$$

25. $$\cos\,(\omega t \pm \phi)\mathbf{1} = \frac{p^2\cos\phi \mp \omega p\sin\phi}{p^2 + \omega^2}\mathbf{1}$$

26. $$\epsilon^{-\beta t}\sin\,(\omega t \pm \phi)\mathbf{1} = \frac{p\omega\cos\phi \pm p(p + \beta)\sin\phi}{(p + \beta)^2 + \omega^2}\mathbf{1}$$

27. $$\epsilon^{-\beta t}\cos\,(\omega t \pm \phi)\mathbf{1} = \frac{p(p + \beta)\cos\phi \mp \omega p\sin\phi}{(p + \beta)^2 + \omega^2}\mathbf{1}$$

28. $$\left(\frac{p}{p + 2\alpha}\right)^{\frac{1}{2}}\mathbf{1} = \epsilon^{-\alpha t}\frac{p}{(p^2 - \alpha^2)^{\frac{1}{2}}}\mathbf{1} = \epsilon^{-\alpha t}J_0(i\alpha t)$$

29. $$\frac{p}{(p^2+\alpha^2)^{\frac{1}{2}}}\mathbf{1} = J_0(\alpha t)\mathbf{1}$$

30. $$\frac{p}{(p^2+\omega^2)(p+\alpha)}\mathbf{1} = \left[\frac{t^2}{\underline{|2}} - \frac{t^3}{\underline{|3}} + \frac{t^4}{\underline{|4}}(\alpha^2-\omega^2) - \frac{t^5}{\underline{|5}}\alpha(\alpha^2-\omega^2) + \frac{t^6}{\underline{|6}}(\omega^4 - \alpha^2\omega^2 + \alpha^2)\right]\mathbf{1}$$
$$= \frac{1}{\omega\sqrt{\alpha^2+\omega^2}}[\sin(\omega t - \beta) + \epsilon^{-\alpha t}\sin\beta]\mathbf{1}$$
$$\sin\beta = \sin\left(\tan^{-1}\frac{\omega}{\alpha}\right) = \frac{\omega}{\sqrt{\alpha^2+\omega^2}}$$

Relations Involving Hyperbolic and Allied Functions:

31. $\epsilon^{\pm x} = 1 \pm \frac{x^2}{\underline{|2}} \pm \frac{x^3}{\underline{|3}} \pm \cdots = \cosh x \pm \sinh x$

32. $\epsilon^{\pm jx} = 1 \pm jx - \frac{x^2}{\underline{|2}} \mp \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} \cdots = \cos x \pm j\sin x$

33. $\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \cdots = \frac{\epsilon^x - \epsilon^{-x}}{2}$

34. $\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \cdots = \frac{\epsilon^x + \epsilon^{-x}}{2}$

35. $\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2j}$

36. $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} \cdots = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2}$

37. $\sinh jx = j\sin x$

38. $\cosh jx = \cos x$

39. $\sin jx = j\sinh x$

40. $\cos jx = \cosh x$

41. $\sinh(-x) = -\sinh x$

42. $\cosh(-x) = \cosh x$

43. $\cosh^2 x - \sinh^2 x = 1$

44. $\frac{d}{dx}\sinh ax = a\cosh ax$

45. $\frac{d}{dx}\cosh ax = \alpha\sinh ax$

46. $$\int \sinh ax\,dx = \frac{1}{a}\cosh ax$$

47. $$\int \cosh ax\,dx = \frac{1}{a}\sinh ax$$

48. $$\sinh x = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{\pi^2 2^2}\right)\left(1 + \frac{x^2}{\pi^2 3^2}\right)\cdots$$
$$= x\prod_1^\infty\left(1 + \frac{x^2}{\pi^2 s^2}\right)$$

49. $$\cosh x = \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{\pi^2 3^2}\right)\left(1 + \frac{4x^2}{\pi^2 5^2}\right)\cdots$$
$$= \prod_1^\infty\left[1 + \frac{4x^2}{\pi^2(2s-1)^2}\right]$$

50. $$\sin x = x\left[1 - \left(\frac{x}{\pi}\right)^2\right]\left[1 - \left(\frac{x}{2\pi}\right)^2\right]\left[1 - \left(\frac{x}{3\pi}\right)^2\right]\cdots$$
$$= x\prod_1^\infty\left[1 - \left(\frac{x}{s\pi}\right)^2\right]$$

51. $$\cos x = \left[1 - \left(\frac{2x}{\pi}\right)^2\right]\left[1 - \left(\frac{2x}{3\pi}\right)^2\right]\left[1 - \left(\frac{2x}{5\pi}\right)^2\right]\cdots$$
$$= \prod_1^\infty\left\{1 - \left[\frac{2x}{\pi(2s-1)}\right]^2\right\}$$

52. $$\sinh(x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$$
$$= \sqrt{\sinh^2 x + \sin^2 y}\ \underline{|\phi_1}$$

53. $$\cosh(x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$$
$$= \sqrt{\cosh^2 x - \sin^2 y}\ \underline{|\phi_2}$$

54. $$\sin(x \pm jy) = \sin x \cosh y \pm j \cos x \sinh y$$
$$= \sqrt{\sin^2 x + \sinh^2 y}\ \underline{|\phi_3}$$

55. $$\cos(x \pm jy) = \cos x \cosh y \mp j \sin x \sinh y$$
$$= \sqrt{\cos^2 x + \sinh^2 y}\ \underline{|\phi_4}$$

Where

$$\tan\phi_1 = \pm \coth x \tan y$$
$$\tan\phi_2 = \pm \tanh x \tan y$$
$$\tan\phi_3 = \pm \tanh y \cot x$$
$$\tan\phi_4 = \mp \tanh y \tan x$$

Factorials and Miscellaneous Formulas:

$x =$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$
$\underline{\lvert x} =$	$\sqrt{\pi}\frac{1}{2}$	$\sqrt{\pi}\frac{1}{2}\frac{3}{2}$	$\sqrt{\pi}\frac{1\cdot 3}{2\cdot 2}\frac{5}{2}$	$+\sqrt{\pi}$	$-2\sqrt{\pi}$	$+\frac{2\cdot 2}{1\cdot 3}\sqrt{\pi}$

$$56.\quad (x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{\underline{\lvert 2}}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{\underline{\lvert 3}}x^{n-3}y^3 + \cdots$$

$$p^n(uv) = u + np^{n-1}(v)p(u) + \frac{n(n-1)}{\underline{\lvert 2}}p^{n-2}(v)p^2(u) + \cdots$$

$$\left.\begin{array}{ll} 57. & \sqrt{1 \pm x} = 1 \pm \frac{1}{2}x - \frac{1}{2}\frac{1}{4}x^2 \pm \frac{1}{2}\frac{1}{4}\frac{3}{6}x^3 - \cdots \\ 58. & \frac{1}{\sqrt{1 \pm x}} = 1 \mp \frac{1}{2}x + \frac{1}{2}\frac{3}{4}x^2 \mp \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + \cdots \\ 59. & \frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots \\ 60. & \frac{1}{1 \pm x^2} = 1 \mp 2x + 3x^2 \mp 4x^3 + \cdots \end{array}\right\} x^2 < 1$$

$$\left.\begin{array}{ll} 61. & \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \cdots \\ 62. & \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \\ 63. & \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \\ 64. & \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \end{array}\right\} \text{By Fourier's series}$$

Expansion Theorem:

$$\frac{Y_{(p)}}{Z_{(p)}}\mathbf{1} = \frac{Y_{(o)}}{Z_{(o)}}\mathbf{1} + \sum_{p_1,p_2\,\cdot\,} \frac{Y_{(p)}\epsilon^{pt}}{p\frac{dZ}{dp}}\mathbf{1}$$

$$\frac{1}{H_{(p)}}\mathbf{1} = \frac{1}{H_{(o)}}\mathbf{1} + \sum_{p_1p_2\,\cdot\,\cdot} \frac{\epsilon^{pt}}{pH'_{(p)}}\mathbf{1}$$

"Shifting" Theorem:

$$f(p)(u\epsilon^{\alpha t}) = \epsilon^{\alpha t}[f(p+\alpha)u]$$

(where u is a function of t)

Duhamel's Theorem:

$$f(t) = E_{(0)}\phi(t) + \int_0^t \phi(t-u)\frac{\partial E(u)}{\partial(u)}du$$

$$= E(t)\phi(o) + \int_0^t E(u)\frac{\partial\phi(t-u)}{\partial(t-u)}du$$

$$\epsilon^{-\alpha t}\int_0^t \epsilon^{\alpha t}\sin(\omega t+\delta)dt$$

$$= \frac{\sin(\omega t+\delta-\phi) - \epsilon^{-\alpha t}\sin(\delta-\phi)}{\sqrt{\alpha^2+\omega^2}}$$

$$= \frac{-\cos(\omega t+\delta+\psi) + \epsilon^{-\alpha t}\cos(\delta+\psi)}{\sqrt{\alpha^2+\omega^2}}$$

$$\epsilon^{-\alpha t}\int_0^t \epsilon^{\alpha t}\cos(\omega t+\delta)dt$$

$$= \frac{\cos(\omega t+\delta-\phi) - \epsilon^{-\alpha t}\cos(\delta-\phi)}{\sqrt{\alpha^2+\omega^2}}$$

$$= \frac{\sin(\omega t+\delta+\psi) - \epsilon^{-\alpha t}\sin(\delta+\psi)}{\sqrt{\alpha^2+\omega^2}}$$

$$\left(\tan\phi = \frac{\omega}{\alpha} = \cot\psi\right)$$

Wave Shapes Expressed by Means of Fourier's Series.—(The Gibbs phenomenon is not shown):

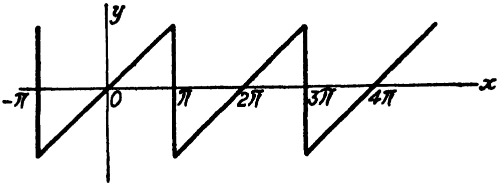

FIG. 55.

$y = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots)$
$y = x$ for $\pi > x > -\pi$

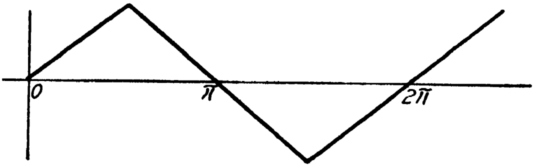

FIG. 56

$$y = \sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x \cdots$$

$$y = \frac{1}{4}\pi x,\ 0 < x < \frac{\pi}{2}$$

$$y = \frac{1}{4}\pi(\pi - x),\ \frac{\pi}{2} < x < \pi$$

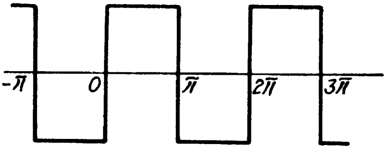

FIG. 57

$$y = 4\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots\right)$$
$$y = \pi,\ 0 < x < \pi$$

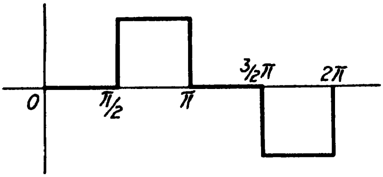

FIG. 58.

$$\begin{aligned} y &= \sin x - \sin 2x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x \\ &\qquad - \tfrac{1}{3}\sin 6x + \tfrac{1}{7}\sin 7x \cdots \\ &= \sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots \\ &= (\sin 2x + \tfrac{1}{3}\sin 6x + \tfrac{1}{5}\sin 10x + \cdots) \end{aligned}$$

$$y = 0,\ 0 < x < \frac{\pi}{2} \quad y = \frac{\pi}{2},\ \frac{\pi}{2} < x < \pi$$

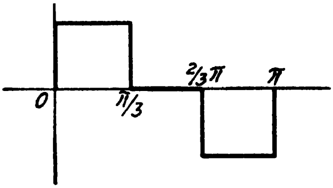

FIG. 59.

$$y = \sin 2x + \tfrac{1}{3}\sin 6x + \tfrac{1}{5}\sin 10x + \cdots$$

$$y = \frac{\pi}{3},\ 0 < x < \frac{\pi}{3};\ y = 0,\ \frac{\pi}{3} < x < \frac{2}{3}\pi;\ y = -\frac{\pi}{3},\ \frac{2}{3}\pi < x < \pi.$$

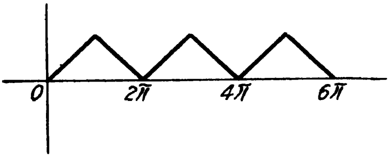

FIG. 60.

$$y = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

$$y = x,\ 0 < x < \pi;\ y = -x,\ -\pi < x < 0$$

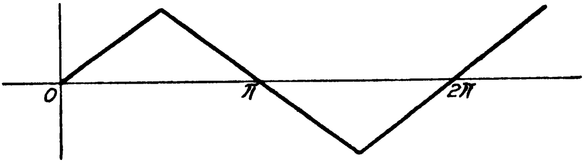

FIG. 61.

$$y = \frac{\pi^2}{16} - 2\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \cdots\right]$$

$$y = \frac{1}{4}\pi x,\ 0 < x < \frac{\pi}{2};\ y = \frac{1}{4}\pi(\pi - x),\ \frac{\pi}{2} < x < \pi$$

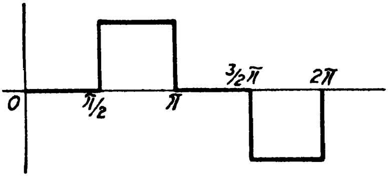

FIG. 62.

$$y = \frac{\pi}{4} - \left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x \cdots\right]$$

$$y = 0,\ 0 < x < \frac{\pi}{2};\ y = \frac{\pi}{2},\ \frac{\pi}{2} < x < \pi$$

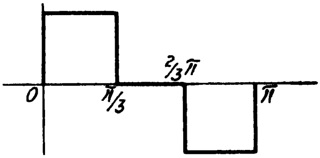

FIG. 63.

$$y = \tfrac{2}{3}\sqrt{3}\ [\cos x - \tfrac{1}{5}\cos 5x + \tfrac{1}{7}\cos 7x - \tfrac{1}{11}\cos 11x + \cdots\ |$$

$$y = \frac{\pi}{3},\ 0 < x < \frac{\pi}{3};\ y = 0,\ \frac{\pi}{3} < x < \frac{2}{3}\pi;\ y = -\frac{\pi}{3},\ \frac{2}{3}\pi < x < \pi$$

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
0 00	1.0	0 25	0 78	0 80	0.449	1 8	0 165
0 02	0 98	0.30	0 741	0.85	0 427	2 0	0 135
0 04	0 96	0.35	0 705	0.90	0.407	2 5	0.084
0.06	0.942	0 40	0 67	0 95	0 387	3 0	0 05
0 08	0.923	0 45	0.638	1 0	0 368	4 0	0 018
0 10	0.905	0 50	0 607	1 1	0 333	5 0	0 0067
0 12	0.887	0.55	0.577	1 2	0 301	6 0	0 0025
0.14	0.870	0 60	0.549	1 3	0 273	7 0	0 0009
0.16	0.852	0.65	0.522	1 4	0.247	8 0	0 00034
0.18	0.835	0 70	0.497	1 5	0.202	9 0	0 00012
0.20	0 819	0 75	0 472	1 8	0 165	10 0	0 00004

APPENDIX

THE WORK OF OLIVER HEAVISIDE[1]

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There is a reference in that beautiful book of Lord Rayleigh's on "The Theory of Sound,"[2] on the subject of electrical vibrations to which we may devote a few moments' consideration. Lord Rayleigh says: "The theory of electric currents in such wires as are commonly employed in laboratory experiments is simple, mainly in consequence of the subordination of electrostatic capacity. When this element can be neglected, the current is necessarily the same at all points along the length of the wire, so that *whatever enters a wire at the sending end leaves it unimpaired at the receiving end.* In this case the whole electrical character of the wire can be expressed by two quantities—its resistance R and inductance L—and *these may usually be treated as constants,* independent of the frequency." For clearness the vital part of the statement is given in italics.

To introduce the reader further into the subject in which the genius of Heaviside has blazed a path of singular clarity and beauty, we shall follow Lord Rayleigh further in his simple presentation rendered in 1894: "Circuits employed for practical telephony may often be regarded as coming under the above description, especially when the wires are suspended and are of but moderate length. But there are other cases in which electrostatic capacity is the dominating feature. The theory of electric cables was established many years ago by Lord Kelvin[3] for telegraphic purposes." A simple review of the mathematical argument will help to refresh the memory, and it is very simply given by Rayleigh. Thus we shall continue: "If S be the capacity and R the resistance of the cable, reckoned per unit length, V and C the potential and the current at the point z, we have

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[1] Reprinted by permission from The Electric Journal for January and February, 1928, Pittsburgh, Pa.

[2] Strutt, John William, Baron Rayleigh, "The Theory of Sound," Vol. 1, p. 466 *et seq.*, London, Macmillan & Co., Ltd., 1894.

[3] The date of Lord Kelvin's paper on "The Theory of the Electric Telegraph" is May, 1855. The paper, written when he was plain Prof. William

$$S\frac{dV}{dt} = -\frac{dC}{dz}, \quad RC = -\frac{dV}{dz} \tag{1}$$

whence

$$RS\frac{dC}{dt} = \frac{d^2C}{dz^2} \tag{2}$$

the well-known equation for the conduction of heat discussed by Fourier."

Let us digress for the moment from Lord Rayleigh's exposition and dwell on the fact that Kelvin's entire cable theory was built upon the conception of "electric diffusion," rather than on a theory of propagation of waves along conductors. Electric waves were, of course, not thought of at the time William Thomson formulated his theory of the electric telegraph, and it is natural enough that he followed Fourier in establishing a partial differential equation for the diffusion of electricity through a cable, an equation the solution of which was rendered possible through Fourier's epoch-making work beginning in 1807 and ending in 1822. Thus, there was a long period of 33 years between Thomson's paper and the mathematical theory which made that paper possible. This is instructive in comparing the development in regard to the mathematical work of Heaviside and its application to engineering problems.

We now continue with the very direct solution of the Fourier partial differential equation (2), assuming the variation of the current, here denoted by C following the earlier custom of the time, to be sinusoidal, expressed operationally by $C = C_0\epsilon^{ipt}$, where ϵ is the Naperian base of logarithms, $p = 2\pi/T$ the angular velocity of the current, T the time of a complete period, $p/2\pi$

Thomson and before a workable transatlantic cable connected this country with Ireland, is well worth careful study even in this day. It is reprinted in Vol. II, p. 61, of *Math. Phys. Papers*, Cambridge, 1884. Lord Kelvin limited himself strictly to distributed resistance and capacity, neglecting inductance.

the frequency, t the time, and $i = \sqrt{-1}$, now frequently designated by j. "On the assumption that C is proportional to ϵ^{ipt}, it reduces to

$$\frac{d^2C}{dz^2} = \left\{ \sqrt{\left(\frac{1}{2}pRS\right)}(1 + i) \right\}^2 C \tag{3}$$

so that the solution for waves propagated in the positive direction is

$$C = C_0\epsilon^{-\sqrt{\frac{1}{2}pRS}\,z} \cos\left(pt - \sqrt{\left(\frac{1}{2}pRS\right)} \cdot z\right) \tag{4}$$

The distance over which the current is attenuated in the ratio of ϵ to 1 is, thus,

$$z = \sqrt{\frac{2}{pRS}} \tag{5}$$

"A very slight consideration of the magnitudes involved is sufficient to give an idea of the difficulty of telephoning through a long cable. If, for example, the frequency $(p/2\pi)$ be that of a note rather more than an octave above middle c, and the cable be such as are used to cross the Atlantic, we have, in c.g.s. measure,

$$\sqrt{p} = 60, \ (RS)^{-1} = 2 \times 10^{16}$$

and, accordingly, from (5),

$$z = 3 \times 10^6 \text{ cm.} = 20 \text{ miles approximately}$$

"A distance of 20 miles would thus reduce the intensity of sound, measured by the square of the amplitude, to about a tenth, an operation which could not be repeated often without rendering it inaudible. With such a cable the practical limit would not be likely to exceed 50 miles, more especially as the easy intelligibility of speech requires the presence of tones still higher than is supposed in the above numerical example.

"In the above theory the insulation is supposed to be perfect and the inductance to be negligible. It is probable that these conditions are sufficiently satisfied in the case of a cable, but in other telephonic lines the inductance is a feature of great importance. The problem has been treated with full generality by

Heaviside . . . " Then Lord Rayleigh refers his readers to Vol. II of Oliver Heaviside's "Electrical Papers," pages 125 and 309, reprinted from *The Electrician*, London, 1886–1887.

The limitations to long-distance telephony had been recognized and analyzed with great skill and acumen by Thomas H. Blakesley, of the Royal Naval College, Greenwich, in a series of papers[1] published in *The Electrician*, October, 1885. This remarkable series of papers treated the problems of series and parallel connection of resistances, inductances, and capacity from the point of view of the modern vector methods. Distributed capacity and inductance are also treated, and most useful hyperbolic tables are given for the calculation of phenomena with distributed constants. Professor Blakesley introduced his papers with a preface from which it is instructive to quote to bring out a useful sidelight on the general status of knowledge in the year 1885. He says: "I have elsewhere strongly disapproved of the action of the Board of Trade in allowing electricity in alternating currents to be distributed on a large scale publicly. Nor must I be held recommending this in the following chapters. But in the last one I have shown how harmonic currents may be advantageously used for measuring some of the electric properties of a circuit and consequently as a means of testing be found extremely useful." He would thus forbid the use of alternating currents. This extraordinary statement gives a good idea of the difficulties of the early pioneers who, like Tesla and Kapp, like Ferranti and Mordey, were about to begin their labors in applying alternating currents to the distribution of power.

Blakesley gives another helpful conception when, on page 63 of the papers already cited, he says of the state of the art of telephony: "Thus at the end of any considerable length and capacity the various tones of the voice would be received in a state of degradation depending upon their pitch. But the ear has not the synthetic power of reconstructing a composite tone from the wreck of variously degraded components. In this consideration lie the limits of telephony. And until it is more clearly understood than it seems to be at present, people will fail to understand the exquisite nonsense to which they are often now content to listen."

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[1] Blakesley, Thomas H., London, "Alternating Currents of Electricity," published at the office of *Electrician*, 1 Salisbury Court, Fleet Street, E. C., 1885.

Thus it required the treatment by Oliver Heaviside of the propagation of electric impulses along wires "with full generality" - to use Lord Rayleigh's happy phrase—to clarify the intricate problems involved in the transmission of speech and power over a long line with the characteristic constants of resistance, inductance, capacitance, and leakance. This work was done in masterly fashion, and its development took place in opposition to the prevailing ideas of the time which, following the methods used in ocean telegraphy, laid great stress upon the electrostatic capacity of the cable and condemned as injurious both the inductance and the leakance of the circuit. As Lord Rayleigh says: "It might perhaps have been expected that a finite leakage would always act as a complication; but Heaviside has shown that it may be so adjusted as to simplify the matter."[1]

These introductory remarks are made to acquaint the reader with the nature of the problem, the discussion of which forms the principal work of Heaviside and, therefore, the substance of this article.

The Historical Setting.—The electric transmission of speech is a process of power transmission. To be sure, the amount of power is small, though metaphorically, the power of speech is great. But though only a small amount of power is transmitted in telephony over a great distance, the process of transmission and the theory thereof are truly identical. Great emphasis should be laid on this statement, and the reader should challenge its truth so as to convince himself that these two processes, seemingly so different, are, in truth, identical.

Thus if the theory of the electric cable is once obtained with "full generality," this theory must explain not merely the vast branch of phenomena of telephony and telegraphy but also the equally vast field of the problems involved in the transmission of power.

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[1] HEAVISIDE, OLIVER, *Electrician* June 17, 1887; *Elec. Papers*, Vol. II, pp. 125, 309; also, his simple and beautiful treatment of the distortionless circuit in "Electromagnetic Theory," Vol. I, pp. 409, 419, 1893, and Vol. II, pp. 291, 1899. The last reference was not available to Lord Rayleigh at the time, 1894, when his "Theory of Sound" was published.

Power transmission and speech transmission hold the stage of discussion on the broadest scale at the moment. Can a time be more suitable, 40 years after the presentation of such a theory, for a review of the work of its foremost protagonist?

The Setting of the Stage.—I am going to take the reader to the year 1887 and set before him in broad outline the condition of electrical engineering at that time. Power engineering and telegraph and telephone engineering had not yet been separated. The American Institute of Electrical Engineers was barely three years old. Maxwell's electromagnetic theory based on Faraday's experimental researches, eliminating the conception of action at a distance, had led him to the prediction of the existence of electromagnetic waves in space propagated with the velocity of light and akin to light except for their greater wave length. The existence of these waves was brilliantly demonstrated by Hertz through his discovery, in 1887, of electromagnetic waves in space and along wires showing all the characteristics of refraction and reflection, only on an infinitely larger scale than light waves. Oliver Lodge had also demonstrated electric waves along wires. These are the waves we are now all familiar with as the waves used in radiotelegraphy and broadcasting.

It is not often in the history of science that so stirring a discovery is made as that of electromagnetic waves predicted in all their characteristics long before their discovery. One has to go back to the finding in its calculated place of the unknown planet Neptune, whose existence and position were predicted in 1846 by Leverrier and Adams, to be found shortly afterward in the predicted place by Professor Galle. Here the discovery followed quickly on the prediction, whereas it took 20 years for the electromagnetic waves to make their appearance after their prediction.

Power transmission was attempted by all means except by electrical means, the exception proving the rule. A little plant of 50 horsepower in Switzerland, operating at 2,000 volts direct current, transmitted this power over a distance of 5 miles. This plant was designed by C. E. L. Brown. Nikola Tesla had not yet made his great inventions. The telephone, invented simultaneously by Alexander Graham Bell and Elisha Gray in 1876, was hardly a decade old.

This is a bare outline of the condition of the electrical field at the time when, in 1887, we become acquainted with Oliver Heaviside.

A Personal Sketch.—The biographical data which are obtainable regarding Heaviside are meager and besides, to a great extent, are of a very personal and, therefore, of a private nature. Born in London on May 13, 1850, Oliver Heaviside died in complete seclusion on Feb. 3, 1925. The last 50 years of his life were spent in this retirement, the greater part at Torquay on the south coast of Devonshire. He was a nephew of Sir Charles Wheatstone, who is known as the father of telegraphy, the inventor of the Wheatstone bridge and of the principle of self-excitation of the dynamo-electric machine. Doubtless Heaviside inherited the gift of physical interpretation from the Wheatstones, on his mother's side. Heaviside's means were very limited, in fact he died in what would be to most of us a state of poverty. It seemed to the present writer that he lacked the necessary care which a man of his age, not to mention his greatness as a mathematical physicist, should have had. It was, however, difficult to render assistance of any kind, as he was a proud man and deeply sensitive. He received a small pension from the British government, which was augmented by an annuity from the British Institution of Electrical Engineers. The writer has given elsewhere a sketch of Heaviside's life.[1]

During the last 10 years of his life Heaviside received numerous proofs of the great respect in which he was held by those who see in the intellectual progress of mankind the principal hope for the future. In 1918, he received one of the two honorary memberships of the American Institute of Electrical Engineers, which usually go to British men of science or engineering. In 1921, he received the first Faraday Medal of the Institution of Electrical Engineers of Great Britain, and the president of the Institution himself journeyed to Heaviside's lonesome country place to present the medal to "one whose genius, perspicacity, and clearsightedness into fundamentals entitle him to be ranked with the greatest of philosophers, Archimedes, Newton, Kelvin, and Faraday." Whether these eulogistic terms are extravagant

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[1] LODGE, SIR OLIVER, and B. A. BEHREND, "Memorial Sketches of Oliver Heaviside," *Elec. World*, Feb. 21, 1925.

—the British are not apt to use superlatives—must be left to the judgment of future generations.

It is, however, pleasant to those of us who were his friends to know that, before he passed away, his work was appreciated on every hand and there was no voice bold enough to deny the greatness and inspiration of this wonderful mind. Well might one have exclaimed on the news of his death, "Let there be silence in the telephone, for Oliver Heaviside has died!"

ELECTROMAGNECTIC INDUCTION AND ITS PROPAGATION

To the engineer it is natural that the subject of the propagation of electromagnetic induction offers the most fruitful field of inquiry. This subject may, therefore, be examined, and a brief analysis, following Heaviside, be given.

In 1887, Heaviside started his series of papers in *The Electrician* on electromagnetic waves.[1] It must be remembered that at the time these papers were written no such waves had been observed, but the prediction which was made by Maxwell was, of course, known to his principal interpreter. Before Hertz discovered Maxwell's waves, Heaviside writes equally prophetically: "When we have little distortion, we get into the region of radiation. The dielectric should be the central object of attention, the wires subsidiary, determining the rate of attenuation. The waves are waves of light, in all save wave length, which is great, and gradual attenuation as they travel, by dissipation of energy in the wires. There is the electric disturbance and the magnetic disturbance keeping time with it, and perpendicular to it, and both perpendicular to the transfer of energy, which is parallel to the wire, very nearly. A tube of energy current may be regarded as a ray of light (invisible, of course).

"It is to such *long* waves that I attribute the magnetic disturbances that come from the sun occasionally and simultaneously show themselves all over the world, arising from violent motions of large quantities of matter, giving shocks to the ether, and causing passage from the sun of waves of enormous length. On such a wave passing the earth, there are immediately induced currents in the sea, earth's crust, telegraph lines, etc."[1] Such reasoning is to us perfectly familiar, but it was Heaviside who made it familiar to us.[2] We shall have occasion frequently to direct the reader's attention to the valuable comments and to the work of such eminent physicists as Lorentz in its bearing on, and connection with, the work of Heaviside. His long struggle for recognition was rewarded a few years before his death by a careful and profound study of his work, not only by the few who had previously gleaned valuable information from it but also by the orthodox school of eminent men of science. It was fortunate that he lived to see his work thus forming the foundation of electrical knowledge, though he had no share in the great material prosperity which it had wrought for the world.

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[1] "Electromagnetic Induction and Its Propagation," *Elec. Papers*, Vol. II, p. 39 (second half), London, Macmillan & Co. Ltd., 1892. The first half appeared in 1885 and following, and it is published in the first volume of the *Collected Papers*. It deals principally with Maxwell's theory.

The problem which occupied Heaviside in the middle 'eighties was then primarily the propagation of electromagnetic force in space and along parallel conductors. The former led him to formulate the fundamental equations of electromagnetic theory in the form of a symmetrical pair of the electric and magnetic forces. These are the circuital equations of Maxwell in their simplified vectorial form, and written by Heaviside

$$\operatorname{curl} \mathbf{H} = \mathbf{J}$$
$$-\operatorname{curl} \mathbf{E} = \mathbf{G}$$

The latter led to the "telephone equation" which I shall discuss at greater length. I propose to take the reader to Heaviside's paper in which he discusses the method of approach which he followed 40 years ago, and which led him to the discovery of the "distortionless circuit" and to the theory of long cables and lines for the transmission of alternating currents.

It is well known how unmanageable the solutions of differential equations appear when general functions are used as solutions. Heaviside recognized this. He says: "In some respects these difficulties are evaded by the consideration of the solution due to a sinusoidal impressed force. The method is very powerful; and, by considering the nature of the result through a sufficiently wide range of frequencies, we may indirectly gain, with comparatively little trouble, knowledge that is unattainable by the method of normal systems."

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[1] *Elec. Papers*, *ibid.*, Vol. II, p. 122.

[2] "That we have got so far is due in the first place to Maxwell, and next to him to Heaviside and Hertz." H. A. Lorentz, "Problems of Modern Physics," a course of lectures delivered at the Calif. Inst. Tech., p. 5, published by Ginn & Company, Boston, 1926.

"But the real desideratum which, if it can be reached, is of paramount importance, is to get solutions which can be understood and appreciated at first sight and followed into detail with ease, presenting to us, as nearly as possible, the effects as they really occur in the physical problem, disconnected from the often unavoidable complications due to the form of mathematical expression. To illustrate this, it is sufficient to refer to the elementary theory of the transmission of waves without dissipation along a stretched flexible cord. If we employ Fourier series, we are doing mathematical exercises. But, use the other method, in which arbitrary disturbances are transferred bodily in either direction at constant speed, *e.g.*,

$$u = f(z - vt),$$

and we get rid of the mathematical complications and can interpret results as we see their physical representatives in reality—for instance, when we agitate one end of a long cord.

"Now there is one case, and, so far as I know at present, only one, in the many-sided question of the transmission of electromagnetic disturbances along wires, which admits of this simple and straightforward method of treatment. Singularly enough, it is not by the simplifying process of equating to zero certain constants, and so ignoring certain effects, that we reach this unique state of things, but rather the other way, generalizing to some extent. It is usual to ignore the leakage of conductors, sometimes also the inductance, and sometimes the permittance [*permittance* is the term used by Heaviside for what he called later *capacitance*]. But we must take all the four properties into account which are symbolized by resistance, leakage conductance, inductance, and permittance, to reach the much-desired result. Briefly stated, the effects are these, roughly speaking: If there be only resistance and permittance, there are, when disturbances of an irregular character are sent along a long circuit, both very great attenuation and very great distortion produced.

The distortion at the end of an Atlantic cable is enormous. Now if we introduce leakage, we shall lessen the distortion considerably but at the same time increase the attenuation. On the other hand, if we introduce inductance (instead of leakage), we shall lessen the attenuation as well as the distortion. And, finally, if we have both leakage and inductance, in addition to resistance and permittance (capacitance), we may so adjust matters, by the effects of inductance and of leakage, being opposite as regards distortion, as to annihilate distortion altogether, leaving only attenuation. The solutions can now be followed nto detail in various cases without any laborious and roundabout calculations. Besides this, they cast much light upon the more difficult problems which occur when not so many physical actions are in question."

"In my usual notation, let R, L, S and K be the resistance, inductance, permittance, and leakage conductance of a circuit, per unit length, all to be treated, in the present theory, as constants; and let E and I[1] be the transverse voltage (this term was also first used by Heaviside) and the current at distance z. The fundamental equations are

$$-\frac{dE}{dz} = (R + Lp)I, \; -\frac{dI}{dz} = (K + Sp)E,$$

p standing for d/dt. Here I is related to the space-variation of E in the same formal manner as E to the space-variation of I. This property allows us to translate solutions in an obvious manner, and gives rise to the distortionless state of things. Let

$$LSv^2 = 1, \text{and } R/L = K/S = q$$

The equation of E is then,—

$$v^2\frac{d^2E}{dz^2} = (q + p)^2E$$

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[1] Heaviside uses V and C for voltage and current but the author has deemed it wise to use the notations to which we are more accustomed in order to show the obvious identity between our modern equations and the historical ones of Heaviside. The latter expressed to E. J. Berg, who called on him a year before his death, a strong dislike for a change in notation. Had the author known this at the time these articles were written, he would have retained Heaviside's notation, as he will in the future.

and the complete solution consists of waves travelling at speed v with attenuation but without distortion. Thus, if the wave be positive, or travel in the direction of increasing z, we shall have, if $f_1(z)$ be the state of E initially,—

$$E_1 = \epsilon^{-qt} f_1(z - vt)$$
$$I_1 = \frac{E_1}{Lv}$$

If E_2, I_2 be a negative wave, travelling the other way,—

$$E_2 = \epsilon^{-qt} f_1(z + vt)$$
$$I_2 = -\frac{E_2}{Lv}*$$

It is now advisable to refer to "Electromagnetic Theory," Vol. I, page 451, 1893, where Heaviside gives the solution of his general equation in the form which has proved best adapted to the purposes of electrical engineering. The equation[1] is familiar to all who are acquainted with power-transmission problems. The e.m.f. necessary to maintain, at the end of a line whose length is x, a current I_0 and a voltage E_0 is

$$E = E_0 \times \cosh qx + I_0 \times Z\frac{\sinh qx}{q} \qquad (10)$$

In this equation, which is the fundamental general solution,

$$q = \sqrt{(r + jL\omega)(k + jS\omega)}$$
$$= \sqrt{ZY}$$
$$Z_0 = \sqrt{\frac{Z}{Y}}$$

Hence, the Heaviside equation (10) can be written

$$E = E_0 \cosh qx + I_0 Z_0 \sinh qx \qquad (11)$$

In this form the equation retains the character of the simple application of Ohm's law by introducing the "surge impedance" which Heaviside—without using the name which was given it later by Professor Kennelly—first introduced to us in *Electrical Papers*, Vol. II, page 369, equation (38), 1887.

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* *Elec. Papers*, Vol. II, p. 308 *et seq.*

[1] The first appearance of this solution is found in *Electrical Papers*, Vol. II, p. 105, Mar. 11, 1887, using trigonometric functions instead of hyperbolic functions which are their equivalents. In view of recent claims, this reference clearly establishes Heaviside as the first to give this form of solution.

To those who have made calculations of long lines, equation (11) is familiar. It is easy to remember and equally easy to see its connection with the nature of the power-transmission problem. The generator voltage equals the receiver voltage plus the drop of potential in the line, exactly as it would be in a direct-current circuit, but both the receiver voltage and the line drop appear multiplied by a function which is a directional quantity or a vector or a complex quantity. This is, of course, obviously necessary by inspection, as the receiver voltage and the line drop must have direction as well as magnitude when dealing with alternating-current circuits.

It is customary to speak of *hyperbolic functions* and their applications to electrical engineering problems.[1] It is, perhaps, a little unfortunate that such names inject a little fear and apprehension into the reader, as he feels that something new and difficult is about to be practiced on him. These functions are simply a convenient grouping of operators of the same nature as the complex quantities, which are, after all is said and done, solely a kind of convenient shorthand to reduce labor and make the clerical part of it less cumbersome. They are interchangeable with the trigonometric forms through simple, elementary, well-known relations. Heaviside used both forms.

THE DISTORTIONLESS CIRCUIT

We were concerned particularly with the conception of the distortionless circuit and its meaning in connection with alternating-current power transmission or speech transmission. It is not easy to follow Heaviside, although intensive study enables one to fall into his mode of working out his problems. It must be remembered that here was a problem baffling the ablest mathematicians, physicists, and engineers and that a novel form of approach had to be designed before the problem would yield to analysis. I will choose here a seemingly different way from Heaviside's, though identical with it and receiving his approval. It is a method suitable to the mode of thought of the engineer and its great simplicity and directness parallel the method used by Heaviside to work out his solution.

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[1] KENNELLY, A. E., "Hyperbolic Functions and Their Applications to Electrical Engineering Problems," 2d Ed., University of London Press, Ltd., London, 1917; "Artificial Electric Lines," McGraw-Hill Book Company, Inc., New York, 1917; NESBIT, WILLIAM. "Electrical Characteristics of Transmission Circuits," 3d Ed , 1926

Heaviside proceeds from the thought that it is necessary, if long-distance telephony is to be made practicable, that the "guides" of the electric propagation—in other words, the line—must have such constants that impulses of all frequencies are propagated at the same speed along these guides and are attenuated alike in transmission. Now, as our quotations from his papers indicate, he felt that it would be easier to obtain this condition while working with four line constants than it would be if he had to work with the two constants used in telegraphy by Lord Kelvin, the resistance and the condensance only. That he solved the distortionless problem before he solved the general problem is instructive.

The object, then, is to obtain waves without alteration of type. Such waves may be attenuated, that is, reduced in geometric proportion, weakening the strength of the signal, but they must not be distorted, that is, part of a wave consisting of a certain frequency must not arrive "out of place" with other parts consisting of another frequency.

In language more familiar to us, the statement can be made as follows: The e.m.f. and the current must retain the phase relation which they have at the sending end. Change in phase relation is distortion. I shall now formulate the problem as follows: To design a circuit through which alternating currents can be transmitted in such a manner that the e.m.f. and the current remain in time phase and attenuating alike, independent of the frequency.

Starting at the receiving end, we assume the e.m.f. and the current of the receiving apparatus to be in phase. Any other relation can be worked out in the same way. Let R be the resistance per unit length, K the leakage conductance, L the inductance, and S the capacitance, all per unit length of the circuit. Now let us build up the voltage and the current from the receiving end in the usual manner by means of e.m.f. and current vectors, keeping, however, before us the condition that the current and the e.m.f. must remain in time phase. This condition will be realized, as is shown at a glance, so long as the e.m.f. and current vectors remain always in time phase, which means that the current and e.m.f. triangles must be similar. This condition prevails, as shown in Fig. 1, when $R/L = K/S = \omega \tan \delta =$ a constant. But this is no other than the Heaviside criterion.

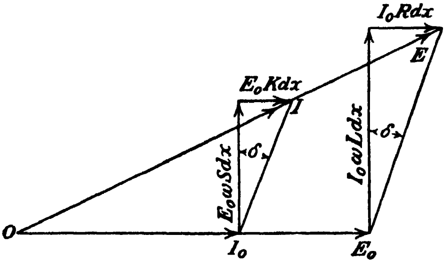

FIG. 1.—Vector diagram for transmission line, currents and voltages being in phase.

But let us go a step farther and interpret Heaviside's solution (10) for this particular case. The differential equation which is the fundamental starting point is

$$\frac{d^2E}{dx^2} = (R + j\omega L)(K + j\omega S)E = q^2E$$

and an identical equation for I.

Let us designate by

$$q = \sqrt{(R + j\omega L)}\sqrt{(K + j\omega S)}$$
$$q = \alpha + j\beta$$

As the product of the "versors"[1] under the radicals must be a complex number or a "vector" quantity—a directional quantity—we have written it in the usual algebraic form $\alpha + j\beta$. Now substitute:

$$R = K\left(\frac{L}{S}\right)$$
$$L = S\cdot\left(\frac{R}{K}\right)$$

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[1] "Really, j is a quadrantal versor, and $a + jb$ and $c + jd$ are also versors with a stretching faculty as well, and their product is another operator of the same sort." "Electromagnetic Theory," Vol. II, p. 458.

Hence,

$$\begin{aligned} q &= \sqrt{(K + j\omega S)\,R/K}\,\sqrt{(K + j\omega S)} \\ &= (K + j\omega S)\sqrt{R/K} = (K + j\omega S)\sqrt{L/S} \\ &= \alpha + j\beta \end{aligned}$$

Hence,

$$\alpha = K\sqrt{\frac{R}{K}} = \sqrt{RK}$$

$$\beta = \omega S\sqrt{\frac{L}{S}} = \omega\sqrt{LS}$$

An inspection of Fig. 1 shows that the real part of q affects the magnitude only of E, and, therefore, it has been designated as the attenuation constant α. It is independent of the frequency. The imaginary part, or the part in time quadrature with the e.m.f., represents a propagation constant β which does not affect the strength of the signal or the current, but which—in a circuit in which there is distortion—would "mix up" the signals, as Heaviside used to say. In the distortionless circuit the velocity of propagation is equal to $1/\sqrt{LS}$ and also independent of the frequency.

Thus, by balancing the leakage of the circuit against the resistance component of the e.m.f. and balancing the charging current against the inductance component, Heaviside devised a circuit over which all signals, whatever their frequency, would travel at the same speed. This must be so, as the phase relations at any part of the circuit between e.m.f. and current remain the same as at the starting point, which is the receiving end in the case under consideration.

For small values of R/L and K/S, as well as RK, a very simple development by expansion of the equation for q can be shown to demonstrate the Heaviside criterion of distortion in which the attenuation depends upon the sum of $R/L + K/S$, and the distortion upon the difference of $R/L - K/S$.

Writing $q = \alpha + j\beta$ and solving the terms under the square root, we find

$$\alpha = \left(\frac{R}{2L} + \frac{K}{2S}\right)\sqrt{LS}$$

and,

$$\beta = \sqrt{\left(\frac{R}{2L} - \frac{K}{2S}\right)\sqrt{LS} + \omega^2 LS}$$

If, now, $R/2L = K/2S$, which is the Heaviside criterion, then the two expressions for α and β become

$$\alpha = \frac{R}{L}\sqrt{LS}$$

$$\beta = \omega\sqrt{LS}$$

$$v = \frac{1}{\sqrt{LS}}$$

$$E = E_0\epsilon^{-\frac{R}{L}\sqrt{LS}\cdot x} \sin\,(\omega t - \omega\sqrt{LS}\cdot x)$$

which is a wave equation in its simplest form indicating that waves of all frequencies are transmitted with the same velocity $v = 1/\sqrt{LS}$. These remarks are helpful in the study of Heaviside's own presentation and are not found elsewhere.

In one of his letters to the present author, Oliver Heaviside proposed the name *heavification* for the process of loading a telephone or telegraph circuit. He suggested that *heavifying* was expressive and it conveyed just credit to its inventor. The adoption of this terminology would be desirable even at this date. The following goes fully into this subject:

The next problem before Heaviside lay in suggesting means for realizing the distortionless circuit in practice. For this purpose, it is necessary to secure a large amount of inductance without undue increase in the resistance. To secure this inductance he suggested two courses. In *Electrical Papers*, Vol. II, 1892, he proposed taking "the finest iron filings [siftings] and mixing them with a black wax," an idea to which he continually reverted in later years. In "Electromagnetic Theory," Vol. I, page 443, 1893, he states: "The above considerations show that if iron be introduced to increase the inductance it should not be in the main circuit but external to it. There are, then, two principal ways suggested: First, to load the dielectric with finely divided iron, and plenty of it. This is very attractive from the theoretical point of view, as it results in the production of a strongly magnetic insulator, which is practically homogenous in bulk, being a sort of non-conducting iron of low permeability. In this way, L, as compared with the same without the iron, may be multiplied greatly." Though it has taken many years to realize the success of Heaviside's "ironic circuit," as he used to call it facetiously, this has at last been done.[1]

Another method for the realization of a distortionless circuit which Heaviside proposed was described by him as follows: "Another indirect way is this: Instead of trying to get large uniformly spread inductance, try to get a large average inductance. Or, combine the two and have large distributed inductance together with inductance in isolated lumps. This means the insertion of inductance coils at intervals in the main circuit. That is to say, just as the effect of uniform leakage may be imitated by leakage concentrated at distinct points, so we should try to imitate the inertial effect of uniform inductance by concentrating the inductance at distinct points. The more points the better, of course. Say m coils in the length 1, or $2m$ coils of the same total inductance, and, therefore, each of half the inductance; or mn coils, each of one nth the inductance of the first. The electrical difficulty here is that inductance coils have resistance as well, and if this be too great the remedy is worse than the disease. But it would seem to be sufficient if the effect of the extra resistance be of minor importance compared with the effect of the increased inductance. This means using coils of low resistance and the largest possible time constants. For, suppose 4 ohms per kilometer is the natural resistance and there be one coil per kilometer having a resistance of 1 ohm. This will raise the average resistance to 5 ohms per kilometer; and if the time constant be big enough, the extra inductance may far more than nullify the resistance evil. The same reasoning applies to coils at greater intervals, only, of course, in a more imperfect manner. To get large inductance with small resistance, or, more generally, to make coils having large time constants, requires the use of plenty of copper to get the conductance and plenty of iron to get the inductance, employing a properly closed magnetic circuit properly divided to prevent extra resistance and cancellation of the increased inductance. This plan does not belong to the category of those mentioned before, which a moment's consideration showed to be worse than useless. It is a straightforward way of increasing *L* largely without too much increase of resistance and may be worth working out and development. But I should add that there is, so far, no direct evidence of the beneficial action of inductance brought in in this way."

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[1] See a paper by Thomas Shaw and William Fondiller, Development and Application of Loading for Telephone Circuits," *Trans. A.I.E.E.*, 1926, p. 268.

The last sentence here quoted, written toward the end of 1893, indicates that the suggestion to try inductance coils in "lumps" about one per kilometer had not been tried at that time. Heaviside had satisfied himself that such a scheme should have practical results, and he appealed to the engineers in the telephone field to try the experiment for verification.[1] Sir William H. Preece, who was at the head of the British telegraph and telephone system, was of the opinion that telephone circuits required more electrostatic capacity for transmission over greater distances and had suggested that the wires be split with insulation between, to increase the capacity by their close juxtaposition. Thus, the ideas of Heaviside fell on deaf ears in his own country. This is very finely expressed in the graceful tribute which Prof. M. I. Pupin paid Heaviside in his remarkable paper before the American Institute of Electrical Engineers describing the first Heaviside loaded line which he devised in his laboratory at Columbia University and described in his paper in May, 1900. The test for which Heaviside had waited 7 years had at last been made in the most successful manner, and it had confirmed for all time the accuracy and the skilful analysis of the theory of the propagation of electromagnetic induction. In a sense, Professor Pupin's confirmation of Heaviside's theory was not unlike the finding, on the part of Professor Galle, of the outermost planet Neptune in the exact position of the heavens where the calculations of the brilliant Leverrier and of Adams had placed it without ever having known of its existence. But we must let Professor

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[1] " . . . they will want to know whether such possibilities can be converted into actualities, and whether the conversion is practicable. For myself, I am not much concerned in this part of the question. It is for practicians to find out practical ways of doing things that theory proves to be possible, or not to find them if they should be impracticable." "Electromagnetic Theory," Vol. I, p. 441.

Pupin voice his own sentiments by giving here a quotation from his paper:[1] "Mr. Oliver Heaviside of England, to whose profound researches most of the existing mathematical theory of electrical wave propagation is due, was the originator and most ardent advocate of wave conductors of high inductance. His counsel did not prevail so much as it deserved, certainly not in his own country."

Pupin's experimental cable was a very beautiful piece of work. It followed the example of Varley and Muirhead in making an artificial laboratory cable or line on which measurements could readily be made which were otherwise difficult to make, as on an Atlantic cable. Thus, in the widest sense, a confirmation of the theories of Heaviside could be carried on, advancing greatly the art of long-distance telephony.

As Professor Pupin pointed out, little recognition had come to Heaviside in his own country. But there was recognition in France at the hands of the chief of the postal system of the fundamental principles laid down by Heaviside. As it is of great historical interest, we here quote from a paper by M. Devaux-Charbonnel, chief engineer of the posts and telegraphs of France.[2] On page 167, the French chief of the postal service remarks that, as early as 1887, Vaschy, then the engineer of the postal service, solved the partial differential equation of electromagnetic propagation and did this independently of Heaviside, whose papers in the *Philosophical Magazine* reached him, however, before the appearance of his own paper, so that he could give Heaviside full credit for his priority. M. Devaux-Charbonnel further states (p. 165) that the French postal administration undertook to make loaded cables of the type of Krarup and Pupin in 1896 and 1897, which in no essential way differed from these cables. Such work was undertaken by M. Barbarat of the French postal service. Thus, we may add the tribute of official France to our own in rendering recognition to the distinguished British scientist.

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[1] *Trans. A.I.E.E.*, Vol. XVII, p. 450, 1900.

[2] Devaux-Charbonnel, Ingénieur en Chef des Postes et Télégraphes. "La Contribution des ingénieurs francais à la téléphonie à grande distance par cables souterrains. Vaschy et Barbarat." *Ann. d. postes, télégraphes et téléphones*, VI année, No. 2, Paris, A. Dumas, Editeur, 1917.

HEAVIFICATION OR THE DISTRIBUTION OF INDUCTANCE IN LUMPS ALONG THE CABLE

The question as to the distance at which such inductance coils for loading the line should be placed has been one of the fascinating aspects of the entire problem. Heaviside suggested, in 1893, that they be placed about one per kilometer so that there should be a sufficient number spread over the shortest wave length to be transmitted. Pupin, in the paper cited, figures the shortest wave length as about 14 miles or 22 kilometers, so that, if Heaviside's suggestion were followed, there would be about 22 coils per wave length. The problem of simulating a uniformly loaded cord by loads placed a certain distance apart and determining the period of vibration of the configuration was proposed first by Lagrange.[1] It is one of the regular exercises in French treatises on dynamics. But it had not yet been applied to the electrical case, and thus Pupin filled in this gap. He found, as was to be expected from the identity of the mechanical and electrical cases, that Lagrange's relation[2]—to wit, the frequencies of vibration—varied as sin $\alpha/2$ to $\alpha/2$ holds also for the electrical case. This relation thus yields readily a comparison between a loaded cable and a uniform cable. In practice the loads are close together and the criterion is invariably fulfilled automatically. A most useful formula for this condition has also been given by G. A. Campbell.[3] This formula proved identical with one obtained for a loaded string by Charles Godfrey.[4] Professor Kennelly has obtained these formulas in a very simple manner in Appendix G of his book "The Application of Hyperbolic Functions to Electrical Engineering Problems," under the heading A Brief Method of Deriving Campbell's Formula. These references have been carefully compiled on account of the difficulty in locating the important papers. It is hoped that they will prove useful, as the study of the papers cited here appears essential to an understanding of the mechanism of the transmission of speech or power over long lines. These papers also crown as a sort of superstructure the work of Heaviside with which we are here primarily concerned.

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[1] LAGRANGE, J. L., "*Mécanique analytique,*" 3d Ed., M. J. Bertrand, Vol. I, Sec. VI, Par. III, No. 50, p. 353, Paris, 1853.

BOUASSE, HENRI, "*Cours de mécanique physique,*" Par. 495, p. 597, 1912.

[2] See, also, TAIT, P. G., "Dynamics," p. 337, 1895: "Waves in a linear system of discrete masses."

[3] In a paper "On Loaded Lines in Telephone Transmissions," *Phil. Mag.*, Ser. VI, Vol. V, p. 313, March, 1903.

[4] *Phil. Mag.*, Ser. XVI, p. 356, 1898. Mr. Campbell states: "An interesting contribution to the general properties of this structure has been made by Mr. Charles Godfrey in a paper on wave propagation along a periodically loaded string and I am indebted to that article for equation (18) which furnishes a complete solution of the propagation."

The groundwork of science grows in a very slow procession. While a great flood of ideas and papers pours forth from the fertile human mind, there are only a few which can be, and should be, made into the groundwork of science. The comprehension of fundamental principles is of the utmost value, not only to the sound progress of science but also to the value of science as a means for the cultivation of the mind. It is on this account that an outline of the theory of electric propagation is given here which can be applied to the solution and understanding of new problems. Heaviside's contributions have been fundamental, and his mode of attack is growing in favor, so that, in a few years' time, our textbooks will present his doctrines, whether under his name or not, to the students of electricity.

MAXWELL THEORY; THE RELATIONS BETWEEN MAGNETIC FORCE AND ELECTRIC CURRENT

The death of James Clerk Maxwell in 1879 had left his magnificent work completed, though in a form which offered formidable analytical difficulties even to such masters as Kelvin, Fitzgerald, Lodge, and Hertz. These were his principal exponents. The difficulty lay, first, in Maxwell's formulation of the two fundamental equations of the electromagnetic field, secondly, in the application of these fundamental equations to the solution of both simple and recondite phenomena.

There is a distinct parallel between the work of Maxwell and the formulation, by Sir Isaac Newton, of the three laws of motion which bear his name. As the whole structure of the science of the motion of particles and rigid bodies is based upon, and clarified by, Newton's three laws, so the science of electromagnetism is based upon the two circuital laws first formulated by Maxwell and thrown into a simple and manageable form by Oliver Heaviside. This achievement required a degree of abstraction, both mathematical and philosophical, in which Heaviside's great mind was aided by the seclusion and solitude in which he spent his life.

As few of our readers are familiar with the conception of a vector and its "curl" and with the "circuital" equations, Heaviside's elementary exposition of these relations is certain to be welcomed. First, a word about the terminology. The reader frequently comes across the words *curl* and *circuitation* in connection with theoretical work in electricity. Maxwell coined the word *curl* to connect the line integral of a vector around a closed curve with the *surface* integral of another vector, which is the *curl* of the first vector, all through the curve. The theorem involved is that known as *Stokes' theorem,* which is fundamental in the derivation of the electromagnetic equations. Lord Kelvin coined the word *circuital*[1] to indicate the annular "portion of the ether uniformly all round its circuit." Thus, the circular lines of force or induction around a long electric circuit are said to be *circuital all round the circuit.* Heaviside gives a really simple method of obtaining the curl of the vector, either of the magnetic force **H** or of the electric force **E**. It may be as well to remind the reader that what we call *magnetomotive force* is the line integral, of the magnetic force **H**; and what we call *electromotive force* is the line integral of the electric force, whence their *curls* can be derived by Stokes' theorem.

We must refer the reader to Heaviside's *Electrical Papers,* Vol. I, page 195, which contains, perhaps, one of the clearest expositions found throughout his writings. Textbook writers would do well to quote this chapter word for word to render the theory in Heaviside's words instead of in their own, as is frequently the practice of writers on dynamics who quote Newton verbatim instead of improving on his wording by using their own. While I must refrain from quoting Heaviside in full, I will give here his introductory remarks on account of their general interest. "Everyone knows that electric currents give rise to magnetic force and has a general notion of the nature of distribution of the force in certain practical cases, as within a galvanometer coil, for example. Further than this, few go. The subject is eminently a mathematical one, and few are mathematicians. There are, however, certain higher conceptions, created mainly by the labors of eminent mathematical scientists, from Ampere down to Maxwell, which are usually supposed to be within the reach of none but mathematicians, but which I have thought could be to a great extent stripped of their usual symbolical dress, and, in their naked simplicity, made to appeal to the sympathies of the many. Let not, however, the reader (if he belong to the many) imagine that thinking can be dispensed with; there is no royal road to knowledge, and hard thinking and rigid fixation of ideas are required. Even the machinery of the mathematician, so great an assistance when made to work, requires severe training on the part of the operator to make it work. But earnest students, if they will not or cannot learn the mathematical methods, need not, therefore, be discouraged, for the name of Faraday will shine forth to the end of time as a beacon of hope and encouragement to them. He was no mathematician, yet achieved results apparently attainable only by such methods. It need not be supposed that he had the peculiar brains of a calculating boy, able to do long sums "in his head" by special methods of his own. The work was of a quite different kind, and probably Faraday could never have made an ordinary mathematician, with the best of training. In fact, mathematical reasoning does not necessarily involve any calculating in the usual sense, though it is, of course, greatly assisted thereby sometimes; and as for the use of symbols, they are merely a sort of shorthand to assist the memory, which even those who openly condemn mathematical methods are glad to use so far as they can make them out—in the expression of Ohm's law, for instance, to avoid spinning a long yarn."

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[1] See Math. Phys. Papers, Vol. III, p. 451..

Heaviside then develops the two equations of Maxwell's in the following form:

First Cross-connection of Magnetic and Electric Force.—This is expressed by

$$\operatorname{curl} \mathbf{H} = 4\pi \mathbf{J}$$

where curl **H** is the vector whose rectangular components are

$$\frac{dH_3}{dy} - \frac{dH_2}{dz}, \quad \frac{dH_1}{dz} - \frac{dH_3}{dx}, \quad \frac{dH_2}{dx} - \frac{dH_1}{dy}$$

and **J** "true" electric current *density*.

But the most useful definition is the following: "The line-integral of a vector **H** around any closed curve or circuit (or the circulation[1] of **H**) equals the surface integral of another vector, *viz.*, "curl **H**," over any surface bounded by the circuit."

Second Connection between Electric Force and Magnetic Force.—The total e.m.f. of induction around a circuit equals the rate of decrease of the amount of magnetic induction through the circuit, or

$$\text{curl } \mathbf{E} = -\frac{d\mathbf{B}}{dt} = -\frac{\mu d\mathbf{H}}{dt}$$

"As the rate of increase of the displacement in a non-conducting dielectric is the electric current, so the rate of increase of $\mathbf{B}/4\pi$ may be called the *magnetic current.* Let it be **G** Then,

$$\mathbf{G} = \frac{d\mathbf{B}}{dt} \cdot \frac{1}{4\pi} \quad \text{(magnetic current per unit area)}$$

Like electric displacement currents, magnetic currents are transient only, *i.e.*, they cannot continue indefinitely in one direction only, like an electric conduction current. Also, like electric currents in a dielectric, they are unaccompanied by heat generation. In ether the electric current and the magnetic current are of equal significance.

"There is probably no such thing as a magnetic conduction current, $g\mathbf{H}$ with dissipation of energy. If there be such, analogous to an electric conduction current, then let

$$\mathbf{J} = k\mathbf{E} + \frac{c}{4\pi} \cdot \frac{d\mathbf{E}}{dt} \quad \text{(true electric current per unit area)}$$

$$\mathbf{G} = g\mathbf{H} + \frac{\mu}{4\pi} \cdot \frac{d\mathbf{H}}{dt} \quad \text{(true magnetic current per unit area)}$$

$$\text{curl } \mathbf{H} = 4\pi\mathbf{J} \tag{15}$$

$$-\text{curl } \mathbf{E} = 4\pi\mathbf{G} \tag{21}$$

where k is the conductance and c is Maxwell's specific inductive capacity of the dielectric or medium.

In this symmetrical form the equations of the electromagnetic field have been written by Heaviside, and this is the form in which they are used today by the foremost physicists of the world. I have not given the full derivation as presented by Heaviside, but I have sketched in rough outline his method and the final form of the fundamental equations. The conception of a magnetic current corresponding with the current in a dielectric is clear and useful from a physical standpoint.

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[1] Later termed *circuitation, E. M. T.*, i, 33.

"When I introduced the new property of matter symbolized by the coefficient *g* [*K* in our notation], it was merely to complete the analogy between the electric and magnetic sides of electromagnetism. The property is non-existent, so far as I know. But I have more recently found how to imitate its effect precisely in another electromagnetic problem, also relating to plane waves . . . " The introduction of the leakage in the conducting medium takes the place of the "magnetic conduction current." "The two dissipations of energy are now due to *R* in the wires, and to *K* in the dielectric, it being *that* in the wires which takes the place of the unreal magnetic dissipation" *Electrical Papers*, Vol. II, p. 379).

It seems certain that Heaviside was led to the conception of the distortionless circuit through his desire to find an analogy in physical phenomena to his fundamental circuital equations. In this endeavor he devised the balance of the leakage dissipation against the resistance dissipation and he thus discovered that though there was no true "magnetic conduction current" in the ether, its effect could be realized artificially by the leakage dissipation in the case of wire waves and he thus invented the distortionless circuit.

MATHEMATICAL METHODS

Heaviside developed two branches of mathematics which are of great value. The first is the method of vector algebra; and the second is the use of "operators" and of divergent series for the solution of unmanageable differential equations.

Wherever directed quantities are encountered, it is common sense to avoid the writing down of lengthy trigonometric equations which become so jumbled that no one knows what to do with them. Vector addition and subtraction are now in general use, but vector multiplication and division as well as extracting roots are really of an elementary character. And yet how few of the readers of this paper are familiar with vector multiplication! To multiply two vectors[1] or complex quantities $A\underline{/\alpha}$ and $B\underline{/\beta}$ means multiplying A and B and adding $\underline{/\alpha}$ and $\underline{/\beta}$. Heaviside's proof was, "It works." He insisted that mathematics was an experimental science and that our vector product above was a sort of shorthand which seemed to work wherever applied. Thus, $A\underline{/\alpha} \times B/\beta = AB(\underline{/\alpha} + \underline{/\beta})$. But other definitions can be given to a vector product. The one at present commonly adopted defines a vector product as a vector at right angles to the original vector planes and of a certain magnitude, as this is useful in electromagnetic theory and in dynamics. There was nothing new in these methods in Heaviside's time except that they fitted the problems to be worked. Every active electrical engineer today uses the complex quantities which Steinmetz and Kennelly have made so familiar. One could not get along without them, and for this Heaviside deserves as much credit as the two able men named.

It was Professor Eddington, himself a famous mathematician, who remarked that a mathematician was never so happy as when he wrote on something he did not understand, and Heaviside likewise talks of mathematicians "inventing difficulties." He also remarks:[2] "An eminent authority once remarked that there is a lot of humbug in mathematical papers. He knew, having done it himself several times." Now, we frequently encounter such papers in modern mathematical applications. Heaviside's interpreters would hardly have been pleased with his own comments on their labors. He disliked the theory of functions; he thought most mathematics dull, with the exception of Fourier. He thought most rigorous proofs silly ("Electromagnetic Theory" Vol. III, p. 140). "It seems to me that the demonstration I have poked fun at is typical of a lot of work made up by the brain-torturers who write books for young people and college students who are going to be senior wranglers, perhaps. Let mathematicians be humanized if possible. The best of all proofs is to set out the fact descriptively, so that it can be seen to be a fact."

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[1] See reference to versors "Electromagnetic Theory," Vol. II, p. 458.

[2] "*Electro Magnetic Theory*" Vol. III, p. 51.

The simplest examples of operators are the complex numbers. We say that the operator j turns a directed quantity from the abscissa or x-axis counterclockwise or positively into the y-axis so that $jx = y$. If you wish to turn it through another quarter phase, multiply it again by j and you have $jjx = j^2 x = -x$. The definition of j is given by $j^2 = -1$ and not by $\sqrt{-1} = j$. Why make a mystery of this? It is shorthand, nothing more or less. But why stop here? Brilliant mathematicians, like the late William Kingdon Clifford, had a way of demonstrating these irrational or complex operations which it may be worthwhile to repeat for the benefit of people who like to know what all this talk about operators really is about. We will begin with Lord Kelvin's famous "compound interest law." If the differential coefficient of a quantity is proportional to the quantity itself, Lord Kelvin would put it that "they follow the compound interest law." Let $y = A\epsilon^{ax}$, then $dy/dx = aA\epsilon^{ax} = ay$. Thus, if we know that a function grows in such a manner that its rate of increase or decrease is proportional to the function itself, then we know that the function is a logarithmic function of the order of y given above.

Clifford went a step farther. If a quantity grows at logarithmic rate jx, he argued, it means what? Write down ϵ^{jx} and interpret it, but how? Algebraically it is equal to

$$\epsilon^{jx} = 1 + jx + \frac{(jx)}{1\cdot 2} + \frac{(jx)^3}{1\cdot 2.3} + \frac{(jx)^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

$$= 1 + jx - \frac{x^2}{2} - j\frac{x^3}{6} + \frac{x^4}{24} -$$

Represent this graphically as in Fig. 2. It shows that, assuming $x = 1$, the operator $\epsilon^{jx} = \epsilon^j$ has turned the step 1 of the above equation counterclockwise through $x = 1$ radians. Give to x different values and the operator keeps on turning the step round and round and round. Hence, it is clear that, as the radius vector OP is equal to $\cos x + j \sin x$, and as OP in magnitude and position is equal to ϵ^{jx}, there follows Euler's important formula

$$\epsilon^{jx} = \cos x + j \sin x$$

From the same figure follows

$$\epsilon^{-jx} = \cos x - j \sin x$$

Hence,

$$2 \cos x = \epsilon^{jx} + \epsilon^{-jx}$$

or

$$\cos x = \tfrac{1}{2}(\epsilon^{-jx} + \epsilon^{jx})$$

If the several terms of the series for cos x be now worked out and constructed step by step, we again obtain a graphical representation of cos x. For $x = 1$,

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \cdots$$

These terms are now all real and are laid off on the abscissa. While these graphical representations may appear too elementary, it is due to the disregard of elementary methods that mathematical analysis has grown to be viewed skeptically.

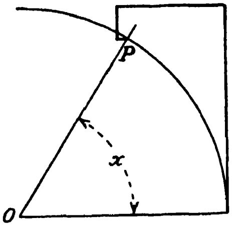

FIG. 2.—A graphical representation of cos x.

These algebraical expressions are totally meaningless without their geometrical interpretation. The moment hard common sense is injected into mathematical argument, it usually appears that a new light and understanding are brought to bear on the task. As algebra is now taught in school, one may be pardoned for agreeing with Heaviside that it is "barbarous" and "inhuman," but it certainly is nothing of the kind in the hands of a John Perry or a William Kingdon Clifford.

The two most important contributions to engineering mathematics made by Heaviside are his chapters On Resistance and Conductance Operators and Their Derivatives, published on pages 355 and 371 of *Electrical Papers*, Vol. II, and The Expansion Theorem. Operational Way of Getting Expansions in Normal Functions, to be found on page 126 of "Electromagnetic Theory," Vol. II.

The last 10 years have seen a great wave of appreciation of these papers now thirty years old. A school has been brought into existence where, under the leadership of Profs. V. Bush and E. J. Berg, a number of postgraduate theses have been based successfully on Heaviside's expansion theorem. At the

Franklin Institute there have been presented recently papers by E. J. Berg, J. J. Smith, Louis Cohen, etc., having for their aim the application of Heaviside's methods to the solution of certain problems in electrical engineering which had heretofore resisted the conventional methods. The American Institute of Electrical Engineers has witnessed a long list of papers by Professors Bush and Vallarta, by Professor Lyon and Dr. Carson and others applying Heaviside's methods to their problems.

The solution of the distribution of e.m.f. and current in alternating-current circuits in the steady state has been made possible for all known configurations by the use of resistance and conductance operators. Solutions of the transient state have been given bv Heaviside, using his expansion theorem, for the sudden application of a constant potential to a circuit. The sudden application of a sinusoidal potential has been worked out, also, in some of the papers referred to. The difficult answers are even more difficult to interpret in the majority of cases which are at all amenable to a solution. Progress in this direction is, perhaps, now more needed in extending the "algebrization"—a term frequently used by Heaviside—of the operational solutions. But in many cases it is very difficult to represent the algebraical solution so that it has any meaning in a physical sense This is a large field in which much work has to be done. Professor Berg's forthcoming book on this subject will be most valuable.

WORK IN OTHER FIELDS

We need not wonder that a genius of the broad interests of Heaviside's type would not have limited himself to the working out of the distortionless circuit and its mathematics, though it would seem a large field, indeed, in view of what has been built upon this foundation. But Heaviside was interested in mathematical physics, and, with his matchless skill in the application of mathematics to this subject, he engaged in numerous critical ventures which may have created the totally wrong idea that he was given to disputation. It is interesting to note that he thus had occasion to point out serious errors made by Lord Kelvin. He pointed out errors in Prof. Peter Guthrie Tait's treatise on quaternions. He analyzed a paper by Sir Joseph J. Thomson, then Professor Thomson, "On Electrical Oscillations in Cylindrical Conductors" and was forced to conclude that "the use of an erroneous boundary condition in the beginning wholly vitiates the subsequent results" (*Electrical Papers*, II, p. 396). His numerous encounters with Sir William Preece, the head of the posts and telegraphs of Great Britain, in which, of course, the late Sir William was invariably wrong, added to the impression that Heaviside was a most formidable opponent in the arena of science. He was always most kind and considerate except in the case of Preece whom he held up to scorn for his lack of understanding combined with great assurance which goes so often with official ignorance. This was, in a sense, unfortunate, at least for Heaviside personally, though it doubtless helped to correct the errors of these eminent men.

In the application of Fourier's theory to a great cosmical example, Lord Kelvin saw an opportunity to calculate the age of the earth from her secular cooling. There were some data available, to be sure very meager, which made it possible to calculate the heat gradient, and from it the time when the center of the earth must have been so hot that all known elements would exist only in a gaseous state. The application of Fourier was beautiful, but the result was disastrous, at least to the geologists and the great school of the followers of Darwin and Huxley and the theory of evolution Lord Kelvin was led to believe that the limiting age of the earth was less than 50,000,000 and more than 10,000,000 years. Professor Huxley protested, somewhat meekly, to be sure, stating that there might be unknown substances in the earth which might lead to a different result. This, of course, has been the conclusion arrived at after the discovery of radioactive substances, but Huxley did not prevail against the most illustrious representative of British science. Then, in 1895, Professor Perry, a close friend of both Heaviside and Lord Kelvin, whose assistant he had been, proposed to Heaviside solving the Fourier equation for rocks with different conductivities in series, so to speak. Conventional mathematics could not solve this problem but Heaviside succeeded in solving it and in proving that the age of the earth could thus be easily a thousand times greater than Lord Kelvin's estimate. Sir Ernest Rutherford and his school have since added any number of years that might be required by geologists or biolgists to meet their theories. Both the Kelvin theory and the Heaviside-Perry enlargement thereof are now useful applications of mathematics to physical problems.

It is interesting to allude to another example where Lord Kelvin and Tait made an error in their monumental work on "Natural Philosophy." This was pointed out by Heaviside in *Electrical Papers,* Vol. II, page 245, and refers to the treatment by the illustrious authors of the St. Venant problem of torsion. Such corrections, if viewed impersonally, become of the utmost interest and value to the cause of science, as they show the fallibility of the greatest of our leaders and the necessity for free discussion untrammelled by the ignorance and narrow-mindedness of those whom we may call with Herbert Spencer the "must-be-rights." The errors of Newton and Lagrange, to mention just two great names, have done much harm until resolute criticism brought them to the bar. Thus let it always be! No man living or dead should be exempt from criticism so long as such criticism is based on fact and fairness.

Heaviside became greatly interested in the discovery, in 1897, of the "corpuscle" by Sir J. J. Thomson. This is now called the *electron,* after the example by Johnstone Stoney. It appeared to him that the mass of a moving particle, when the velocity approached that of light, would be a function of this velocity. He developed the equations which are now fundamental.

In the words of Sir Oliver Lodge, Heaviside left his name on the atmosphere, as it was he who first called attention to the existence of an ionized layer in the atmosphere which acted as a reflecting medium for the Hertzian waves. In his own words: "Sea water, though transparent to light, has quite enough conductivity to make it behave as a conductor for Hertzian waves, and the same is true in a more imperfect manner of the earth. Hence, the waves accommodate themselves to the surface of the sea in the same way as waves follow wires. The irregularities make confusion, no doubt, but the main waves are pulled round by the curvature of the earth and do not jump off. There is another consideration. There may possibly be a sufficiently conducting layer in the upper air. If so, the waves will, so to speak, catch on to it more or less. Then the guidance will be the sea on one side and the upper layer on the other."[1]

This narrative makes no attempt at rendering a complete picture of the work during a very long life of one of England's most powerful interpreters of physical phenomena. But it would be even more woefully incomplete if, as a fitting postscript, it were to omit a recital of many of the happy and humorous thoughts and ideas which brighten the 2,700 pages, in fine print and rife with mathematical formulas. Readers would have ground for not forgiving me if I were to withhold from them gleanings so interesting and so picturesque and clear that even those who care nothing for mathematical treatment of physical phenomena would enjoy these five volumes. James Swinburne, reviewing in *Nature* for Dec. 20, 1894, the first volume of the "Electromagnetic Theory," says, toward the close: "The style is that of Whitman, except that Mr. Heaviside is not affected and has something to say. The similarity is also noted in the *Philosophical Magazine*. Every line of the book is important, and it is full of interesting digressions on all sorts of subjects. Though the converse may not be true, all clever men have a sense of humor, and it is therefore a pity that scientific writers emulate the ponderous dryness of the theologian. Mr. Heaviside's work bristles with humor of a type which he has invented."

As readers might like to check quotations from Heaviside's works, I shall designate the *Electrical Papers* by E. P. and the "Electromagnetic Theory" by E. M. T., and our first quotation shall be from Storage Room Is Too Valuable (E. M. T., II, 433). "For the benefit of the uninitiated, I should explain that El. Pa. means my own *Electrical Papers*. They can be picked up cheap, because the remainder was sold off in quires for a few pence per volume, on account of the deficiency in storage room. Though somewhat vexed at first by this disposal of my labored lucubrations, it has later, given me and others occasion for much laughter" (E. M. T., III, 93). "NOTE (Nov. 30, 1887).—The author much regrets to be unable to continue these articles in fulfilment of Sec. XI, having been requested to discontinue them" (E. P., II, 155).

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[1] Electromagnetic Theory," Vol. III, p. 335, June, 1902; see, also KENNELLY, A. E., *Electrical World and Engineer*, Mar. 15, 1902, p. 473, where a prior announcement is made of the probable existence of such a layer.

Now these are the papers which have since been unobtainable. They have sold at $75 for the two volumes and they made desirable a photographic reprint which is now available. It is certainly a bit of humor that the inventor of long-distance telephony, while he was explaining his invention, was asked by his publishers to stop. Fortunately for the telephone, the gag was removed after a little time.

Under perpetual irritation from the leading mathematical rigorists of Cambridge, who did not read his work but condemned it, he wrote, "Whether good mathematicians, when they die, go to Cambridge, I do not know" (E. M. T., III, 175). In discussing Lagrange's equations and the principle of least action, he says, "Truly, I have never practiced it myself (except with pots and pans) . . . " (E. M. T., III, 175). "If it is love that makes the world go round, it is self-induction that makes electromagnetic waves go round the world" (E. M. T., III, 194). In regard to Lagrange's equations, he says: "Is not Newton's dynamics good enough? Or do not the Least Actionists know that Newton's dynamics, *viz.*, his admirable Force = Counterforce and the connected Activity principle, can be directly applied to construct the equations of motion in such cases as above referred to, without any of the *hocus-pocus* of departing from the real motion, or the time integration, or integration over all space, and with avoidance of much of the complicated work? It is against its misuse that I write" (E. M. T., III, 176). "When I was a young child I conceived the idea of infinite series of universes, the solar system being an atom in a larger universe on the one hand, and the mundane atom a universe to a smaller atom, and so on" (E. M. T., III, 172). " . . . Faraday . . . that great genius had all sorts of original notions wrong as well as right and, not being a mathematician, could not effectually discriminate . . . " (E. M. T., I, 415). "Some do not believe in the materiality of the ether. This view is thoroughly anti-Newtonian, anti-Faradaic, and anti-Maxwellian. What mean action and reaction, the storage of energy, the transit of force and energy through space, etc., if there is no medium in space? For space is nothing at all, save extension. Lord Kelvin used to call me a nihilist. That was a great mistake, (though I did throw a bomb occasionally, to stimulate an official humbug to learn something about electricity and how to apply it) . . . I have observed repeatedly that young to-be physicists, when they leave college, are full of generalized coordinates, and the theory of functions, and unnatural spaces, and rigor; but when they become thoroughly immersed in real physics, even though mathematical, a lot of the learning referred to fades away. Is Cambridge to blame? Perhaps not. It may be my fault" (E. M. T., III, 479).

"They are purely plain waves . . . They are merely guided through space in a definite manner by the conductors, imagined to have no resistance, so that, to use a very gross simile, the electricity slips along like greased lightning" (E. P., II, 140). "Infinity immeasurable but not un-measurable" (E. M. T., II, 113). "Self-induction is salvation" (E. M. T., II, 354). "Physics is above mathematics and the slave must be trained to suit the master's convenience" (E. M. T., II, 414). "Most solutions of problems in mathematical physics are in the form of infinite series. Finite solutions are quite exceptional" (E. M. T., II, 66). "Let mathematics be humanized, if possible. The best of all proofs is to set out a fact descriptively so that it can be seen to be a fact" (E. M. T., III, 140). "My authority for Newton is that stiff but thorough-going work, Thomson and Tait" (E. M. T., III, 178). "There is room in the ether for much speculation" (E. M. T., III, 98). "But I think a rigid and incompressible ether is an exceedingly difficult idea" (E. M. T., III, 145). "Lord Kelvin's thermoelectric theory has always appeared to me one of his best works" (E. M. T., III, 183). "But perhaps like the fishes who were preached to by the saint, 'Much edified were they but preferred the old way' " (E. M. T., III, 291).

Excerpts from Heaviside's Letters to the Author.—"I think honors have been very much overdone. The more honors the less value. It is depreciating the currency. Of late years there has been a perfect flood of new honors, and even the women have caught the plague."

"If I were offered a Dukedom, I might take it, having already an estate, properly mortgaged up to its full capacity, and I should think I was doing the tribe of Dukes an honor by joining it."

"You do not take notice of my 'heavify' and 'heavification' idea. Perhaps you thought it only a joke."

"But there is rather a funny notion prevalent that an invention is not an invention unless it is patented, and then it is the patentee's invention."

"I want to get on with volume 4 of E. M. T. and may be able to do it bye and bye. It is a question of health and freedom from disturbance and petty worries . . . It is slow work for an old man in any case."

The war came and with it much hardship and suffering for the aged scholar. Let us keep the gaze of the world from the last years of the master. At last the great light went out. *Sic pereat gloria mundi!*

BIBLIOGRAPHY

Berg, E. J.: "Heaviside's Operators in Engineering and Physics," *J. Franklin Inst.*, vol. 198, pp. 647–702, November, 1924.

———, S. J. Haefner, and J. J. Smith: "Asymptotic Solution of Heaviside's Fractional Operators," *J. Franklin Inst.* February, 1928.

Bernstein and Doetsch: "Problem aus der Theorie der Warmeleitung," *Math. Zeit.*, 22, 1925, p. 284.

Bloemsma, J., and C. R. Burch: "On an Application of the Periodogram to Wireless Telegraphy," *Phil. Mag.*, vol. 49, February, 1925.

Breisig, F.: "Theoretische Telegraphie."

Bromwich, T. J. I'A.: "Examples of Operational Methods in Mathematical Physics," *Phil. Mag.* vol. 37, pp. 407–419.

———: "Normal Coordinates in Dynamical Systems," *Proc. London Math. Soc.*, series 2, vol. 15, pp. 401–448, 1916.

———: "Symbolical Methods in the Theory of Conduction of Heat," *Proc. Cambridge Phil. Soc.*, vol. 20, part 4, pp. 411–427, 1921.

Burch, C. R. and J. Bloemsma: "On an Application of the Periodogram to Wireless Telegraphy," *Phil. Mag.* vol. 49, February, 1925.

Bush, V.: "Note on Operational Calculus," *M.I.T. Research Bull.* 41.

———: "Oscillating-current Circuits by the Method of Generalized Angular Velocities," *J. Am. Inst. Elec. Eng.*, vol. 36, pp. 207–234, 1917.

———: "Transmission Line Transients," *M.I.T. Research Bull.* 40.

———: "Operational Circuit Analysis," John Wiley & Son, Inc., 1929.

Campbell, G. A.: "Physical Theory of the Electric Wave Filter," *B.S.T.J.* vol. I, November, 1922.

———: "Electric Wave Filters," Patents 1227113 and 1227114.

———: "Cisoidal Oscillations," *Trans. Am. Inst. Elec. Eng.*, vol. 30, pp. 873–909, 1911.

———: "On Loaded Lines in Telephonic Transmission," *Phil. Mag.*, March, 1903.

Carson, J. R.: "The Heaviside Operational Calculus," *B.S.T.J.*, November, 1922.

———: "The Heaviside Operational Calculus," *Bull. Am. Math. Soc.*, vol. 31, No. 1, January-February, 1926.

———: "On a General Expansion Theorem for the Transient Oscillations of a Connected System," *Phys. Rev.*, vol. 10, pp. 217–225, September, 1917.

———: "Selective Circuits and Static Interference," *B.S.T.J.*, vol. 4, No. 2, pp. 265–279, April, 1925.

———: "Theory of the Transient Oscillations of Electrical Networks and Transmission Systems," *J. Am. Inst. Elec. Eng.*, vol. 38, pp. 345–427, 1919.

Carson, J. R: "Theory and Calculation of Variable Electric Systems," *Phys. Rev.*, vol. 17, pp. 116–134, February, 1921

———. "Building Up of Sinusoidal Currents in Long Periodically Loaded Lines," *B.S.T J.*, vol 3, No 4, October, 1924

———: "Die Behandlung der Telegraphengleichung (auch unter Berucksichtigung der Stromverdrangung) nach der Operatorenmethode," *E N T.*, Band 2, Heft 11, 1925.

——, and O J Zobel: "Transient Oscillations in Electric Wave Filters," *B.S T J*, June, 1923.

———: "Electric Circuit Theory," McGraw-Hill Book Company, Inc., 1926.

Casper, L.: "Zur Formel von Heaviside fur Einschaltvorgange," *Archiv. Elektrotechnik*, vol 15, p. 95, 1925

David, Pierre: "Essai sur la Théorie des Filtres Électriques," *L'Onde Electrique*, p. 5, January, 1926; p 72, February, 1926

Drysdale, C. V.: "The Theory of Alternate Current Transmission in Cables," *Electrician*, vol. 60, pp. 277–279; 316–319; 358–359; 392–394; 468–469.

Evans, G. C: "Functionals and Their Applications," Am. Math Soc, *Cambridge Colloquium*, vol. 5, 1918; "Discontinuous Dirichlet and Neumann Problems," Am. Math Soc., *Colloquium Publications*, 6, 1927.

Fry, T. C.: "The Solution of Circuit Problems," *Phys. Rev*, vol. 1, pp 115–136.

———: "The Application of Modern Theories of Integration to the Solution of Differential Equations," *Ann Math*, vol 22, No 3, March, 1921.

Giorgi, G: "Sul Calcolo delle Soluzioni Funzionali Originate dai Problemi di Elettrodinamica," Associazione Elettrotecnica Italiana, *Estratto daglio Atti*, vol. 9, pp. 651–699, 1905.

———: "Ill Metodo Simbolico nello Studio delle Correnti Variabili," Associazione Elettrotecnica Italiana, *Estratto daglio Atti*.

Heaviside, O: "Electromagnetic Theory," vols 1, 2, 3.

Herlitz, Ivar: "Erzeugung von Stationaren Magnetfeldern in einem Elektrisch Leitenden Material," *Arkiv. Mat., Astron. Fysik*, vol. 14, Nov. 22, 1919.

Johnson, K. S.: "Transmission Circuits for Telephonic Communication."

———, and T E. Shea: "Mutual Inductance in Wave Filters with an Introduction on Filter Design," *B.S.T.J.*, January, 1925.

Kupfmüller, K.: "Störungsverminderung durch selektive Shalt-mittel beim drahtlosen Empfang," *E N.T.*, Band 3, Heft 3, 1926.

———: "Über Einschurnvorgänge in Wellenfiltern," *E.N.T.*, Band 1, Heft. 5, 1924.

Lange, Ch.: "Théorie des Filtres Electriques," *Ann. des Postes, Télégraphs et Téléphones*, pp. 1256–1292, October, 1923.

Levy, P.: "Lecons d'Analyse Fonctionnelle," Gauthier, Paris, 1922.

Malcolm, H. W.: "Theory of the Submarine Telegraph."

Moeller, F.: "Die Abflachung steiler Wellenstirnen unter Berücksichtigung der Stromverdrangung im Leiter," *Archiv. Elektrotechnik*, vol. 15, 6 Heft, 1926.

Nichols, H. W.: "Theory of Variable Dynamical Electrical Systems," *Phys. Rev.*, vol. 10, No. 2, August, 1917.

Peters, L. J: "Theory of Electric Wave Filters Built Up of Coupled Circuit Elements," *Trans. Am. Inst. Elec. Eng.*, vol 62, p. 98, 1923.

Picard, E: "Sur la Distribution de L'Electricite avec la Loi de Neumann et sur la Theorie Analytique de la Chaleur," *Rendiconti del Circolo Matematico Palermo*, 37, 1914, p. 249.

Pierce, G. W.: "Electric Waves," or "Electric Oscillations and Electric Waves," book 2, Chaps. I–IX, pp. 347–434.

Pleijel, H.: "Allmanna egenskaper hos ett System Parallella Ledningar med Variabla Konstanter," Skrifter Utgivna med Anledning av Inflyttningen i de Ar, 1917, Fardiga Nybyggnaderna. (Reprint.)

———: "Om Beräkning af Öfverspanningar," *Teknisk Tids*, pp. 50–56, April, 1914.

———: "Strom och Spanning hos ett System Parallella Ledningar," *Teknisk Tids.*, pp. 105–108, July, 1919.

———: "Vandringsvågor och deras Formförandringar under Fortplantningen Utefter Ledningan," *Teknisk Tids.*, pp. 129–137, November, 1918

Pleijel, H, and R. Liljeblad: "Operatorkalkylens Samband med den Symboliska Metoden," *Teknisk Tids.*, pp. 25–33, February, 1919.

Pollaczek, F.: "Das Einschaltproblem fur homogene Kabeln bei beliebiger Endschaltung," *E.N.T*, Band 1, Heft 3, 1924.

———: "Theorie der Einschaltvorgange des vielgliedrigen Künstlichen Kabels," *E N.T.*, Band 2, Heft 7, 1925.

Pomey, J. B.: "Le Câlcul Symbolique d'Heaviside," *Rev. gén élec.*, vol 13, pp. 813–815, 860–863, May, 1923.

———: "Analogies Mecaniques de L'Électricité," *Bibliotheque des Ann, des Postes, Télégraphes et Tèléphones*, 1921.

Pupin, M. I.: "Propagation of Long Electric Waves," *Trans Am. Inst. Elec. Eng.*, p. 93, 1890.

———: "Wave Propagation over Non-uniform Cables," *Trans Am Inst Elec. Eng.*, p. 445, 1900.

Salinger, H.: "Die Heavisidesche Operatorenrechnung," *E N.T.*, vol. 2, pp. 365–376, November, 1925.

Siegbahm, Manne: "Elektromagnetische Wellen in Spulen, mit Windungskapazitat." *Archiv. Elektrotechnik*, vol 4, pp. 305–313, 1916

Smith, J. J.: "An Analogy between Pure Mathematics and Operational Mathematics of Heaviside by Means of the Theory of H-functions," *J. Franklin Inst.*, pp. 519–534, October, 1925; pp. 635–672, November, 1925; pp. 775–814, December, 1925.

———: "The Solution of Differential Equations by a Method Similar to Heaviside's," *J. Franklin Inst.*, vol. 195, pp. 815–850.

Steinmetz, C. P.: "Theory and Calculation of Transient Electric Phenomena and Oscillations." McGraw-Hill Book Company

Steinmetz, C. P.: "Theory and Calculation of Electric Circuits," *Proc. Am. Inst. Elec. Eng.*, March, 1919.

Vallarta, M. S.: "Heaviside's Proof of His Expansion Theorem," *Proc. Am. Inst. Elec. Eng.*, vol. 65, pp. 383–387, April, 1926.

Wagner, K. W.: "Elektromagnetische Ausgleichsvorgange in Freileitungen und Kabeln."

———: "Über eine Formel von Heaviside zur Berechnung von Einschaltvorgange," *Archiv. Elektrotechnik*, vol 4, p. 159, 1916.

———: "Der Satz von der Wechselseitigen Energie," *E.N.T.*, vol 2, pp. 376–392, November, 1925.

———: "Elektrische Kettenleiter und ihre Technischen Anwendunge," *E.T.Z.*, vol. 42, pp. 1290–1291, Nov. 10, 1921.

———: "Theorie des Kettenleiters," *Archiv. Elektrotechnik*, p 315, 1915.

Wiener, N.: "The Operational Calculus," *Math. Annalen*, vol 95, Heft 4; also *M.I.T.*, series 2, No. 109.

Zobel, O. J.: "Theory and Design of Uniform and Composite Electric Wave Filters," *B.S.T.J.*, vol. 2, January, 1923

———: "Transmission Characteristics of Electric Wave Filters," *B.S.T.J.*, October, 1924.

———, and J. R. Carson: "Transient Oscillations in Electric Wave Filters," *B.S.T.J.*, June, 1923.

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